Digits and Bases Handout

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1 Introduction

1.1 Digits

Numbers are written as series of *digits*, such as 4, 32, 1735, and so on. We can expand these numbers; for example, $1735 = 1 \cdot 1000 + 7 \cdot 100 + 3 \cdot 10 + 5$. Let's see how this basic decomposition lets us solve some basic math problems.

Example 1.1.1. Prove that if a number's digits sum to a multiple of 9, it is divisible by 9.

Solution: First, we can represent any number as $x_n \cdot 10^n + x_{n-1} \cdot 10^{n-1} + \cdots + x_0$. Let's consider the remainder when we divide this number by 9.

For any number 10^n, one can see that the number $10^n - 1$ will be $999 \cdots 999$, where there are n 9s. Obviously, this is a multiple of 9. So when you divide 10^n by 9, the remainder will be 1.

If you multiply 10^n by some number a, the remainder will be $1 \cdot a = a$. This means that when you divide $x_n \cdot 10^n + x_{n-1} \cdot 10^{n-1} + \cdots + x_0$ by 9, the remainder reduces to $x_n + x_{n-1} + \cdots + x_0$! If this sum is a multiple of nine, then we can further reduce the remainder to 0, meaning that we have proven that when a number's sum of digits is a multiple of nine, it itself is also a multiple of 9.

Exercise 1.1.1. Prove the divisibility by 3 rule: if a number's digits sum to a multiple of 3, that number is also an multiple of three. *Hint:* Start with the divisibility by 9 proof; how can you modify it?
Exercise 1.1.2. something
Exercise 1.1.3. something else

1.2 Bases

Recall our expansion of numbers. When we decomposed 1735 into $1000 + 700 + 30 + 5$, we were using powers of 10. However, there's no reason why 10 is special; we can use any integer as the power. For example, 57 is equivalent to $2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$. This brings us to the following notation:

Definition 1.2.1. The number $a_n a_{n-1} a_{n-2} \cdots a_0$ in base k is $a_n \cdot k^n + a_{n-1} \cdot k^{n-1} + \cdots + a_1 \cdot k^1 + a_0$.

We can convert from any one base to another.
Example 1.2.1. Convert 57_{10} into base 2.

Solution: When converting between bases, it helps to know the powers for a base. In this case, we know that 32 is 2^5, so we have our first digit: 1. $57 - 32 = 25$, and the next largest power of 2 is 16, so our next digit is also an 1. $25 - 16 = 9$, so we can subtract 2^3 to get 1. We can't subtract 4 or 2, so those place values are left as 0, leaving the ones place as 1. $57_{10} = 111001_2$.