Iowa City Math Circle Handouts
Coordinate Systems

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1 The Cartesian Plane

It is said that the French mathematician Renee Descartes was a sick boy who spent much of his time in bed. As he lay, he watched a fly move across the ceiling of his bedroom. Descartes realized that no matter where the fly was at, the position could be described as a certain ordered pair based off of the fly's distance from the sides of the ceiling. From here, the Cartesian plane, named after Descartes himself, was derived.

The Cartesian Plane is a grid defined by two perpendicular lines, our x and y axes, where every point on the plane can be represented by a unique ordered pair (x, y), which are defined by the point's distance from the x and y axes. For example, to get to the point $(3, 5)$, we'd start at the origin, $(0, 0)$ and then from there we would move 3 units to the right and then 5 units up. Alternatively, we could move 5 units up and then 3 units to the right, which would still get us to the same point.

2 Lines

In coordinate geometry, a line can be defined by two points on the line - in other words, any two distinct points in the Cartesian plane uniquely determine a specific line. Main characteristics of a line are the x and y-intercepts(where the line hits the x and y axes) and the slope of the line(The ratio of change in y over change in x in the line). There are three main ways to represent a line in coordinate geometry -

- (Point-slope form) $y_2 - y_1 = m(x_2 - x_1)$
- (Standard form) $ax + by = c$
- (Slope-intercept form) $y = mx + b$

Where in point-slope form and in slope-intercept form, the slope of the line is m.

Checkpoint 2.1. Prove that the slope of $ax+by=c$ is $-\frac{a}{b}$ by rearranging the equation into slope-intercept form.

3 Coordinate Geometry Basics

Example 3.1. Let $ABCD$ be a rectangle with $AB = 1$ and $BC = 2$. In addition, let E be a point on diagonal BD such that E trisects the diagonal and is closer to B than D. Find the length of line segment AE.

First, let's attempt to solve this problem synthetically - without coordinates.

Checkpoint 3.1. Solve the above problem.

Now we can use coordinate geometry to attempt to solve the problem .

Solution. We can plot this rectangle on the coordinate plane with A at the origin, $B = (0,1)$, $C = (2,1)$, $D = (2,0)$. Since E trisects BD and is closer to B than D, we have $E = \frac{2}{3}(0,1) + \frac{1}{3}(2,0) = (\frac{2}{3}, \frac{2}{3})$. Using the distance formula or a $45-45-90$ triangle, we have that $\boxed{AE = \frac{2\sqrt{2}}{3}}$.

For simpler problems that have simpler synthetic solutions, coordinate geometry may not be the best way to go as it takes many equations and much patience to bash such computations out. However, as we get to harder problems of which are complicated when attempted synthetically, we may find that the computational coordinate solution is much neater and less time consuming. Another thing to note when solving problems with the Cartesian plane is that it may make your computations a lot easier if you choose your points in a way such that your coordinates are cleaner with more convenient numbers- an example of such is in the previous problem, where we chose A to be the origin for simplified calculations since calculating the distance of any point from the origin is more straightforward than calculating the distance from another point.

4 Some Basic Theorems

The following theorems are the most basic theorems in coordinate geometry and mostly carry the entire subject - once you understand these theorems along with a few techniques, you're all set to solve any coordinate geometry problem.

- Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 = -\frac{1}{m_2}$
- Two lines with slopes m_1 and m_2 are parallel if and only if $m_1 = m_2$

- (Distance Formula) The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- (Midpoint Theorem) The midpoint of the two points (x_1, y_1) and (x_2, y_2) is the point $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$
- (Perpendicular Bisector) If A and B are two points in the plane, the locus of points that are equidistant from the two points is the line that is perpendicular to AB and passes through the midpoint of AB - this is called the perpendicular bisector of AB.

Checkpoint 4.1. Prove that all points on the perpendicular bisector are equidistant from A and B.

Checkpoint 4.2. Prove the midpoint theorem by proving that the given point is equidistant from the original two points.

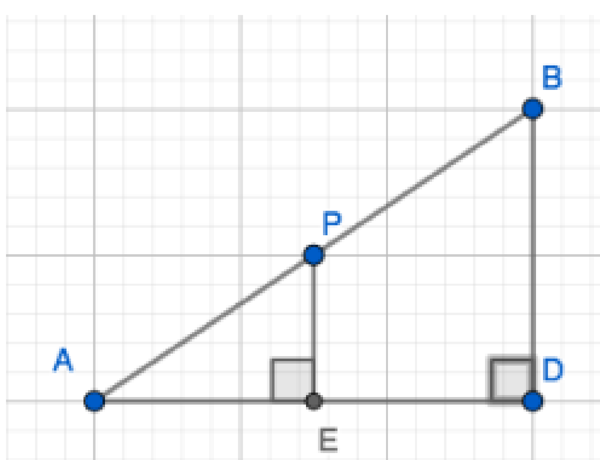
5 Ratio Point Theorem

(Ratio Point Theorem) Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The point P on line segment AB such that the ratio of AP to BP is $1 - r : r$, where $0 < r < 1$, can also be expressed as

$$P = (rx_1 + (1 - r)x_2, ry_1 + (1 - r)y_2) \tag{1}$$

Note that if $r < 1 - r$, then the point P is closer to B than A. Looking at this more generalized version of the midpoint theorem, we can use similarity to prove this.

Proof. It is trivial to see that P lies on the same line segment as AB since AP has the same slope as the line segment AB. Our next step would to then be to prove that $\frac{AP}{BP} = \frac{1-r}{r}$.



Examining the figure, we see that triangles AEP and ADB are similar to each other via AA similarity. Thus the ratio $\frac{AP}{AB}$ will be equal to the ratio

$$\frac{AE}{AD} = \frac{rx_1 + (1-r)x_2 - x_1}{x_1 - x_2} = \frac{(1-r)(x_1 - x_2)}{x_1 - x_2} = 1 - r = \frac{AP}{AB} \tag{2}$$

Meaning that $\frac{AP}{BP} = \frac{1-r}{r}$ and we are done.

Checkpoint 5.1. Find the point that is $\frac{1}{3}$rd of the way from A to B, given that $A = (1, 4)$ and $B = (7, 13)$.

6 Distance From a Point to a Line

(Distance From Point to Line) The distance from a point $A(x_1, y_1)$ and the line $ax_2 + by_2 + c = 0$ has length of

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \tag{3}$$

This can be proved in many ways, using the distance formula and constructing auxiliary lines. Note that the point that which of the distance from the point to the point on the line is the point X such that AX is perpendicular to the given line. This in turn is also the point on the given line that is closest to the point A.

Checkpoint 6.1. Find the distance from the point $(3, 4)$ to the line $5x + 12y = 60$.

7 Shoelace Theorem

The Shoelace Theorem is a very useful theorem in coordinate geometry when it comes to find the area of a n-gon when only given the coordinate of the polygon.

(Shoelace Theorem) The area of an n-sided polygon with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), listed in either clockwise or counter-clockwise order, can be given as

$$\frac{1}{2}|(x_1y_2 + x_2y_3 + ... + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + ... + y_{n-1}x_n + y_nx_1)| \tag{4}$$

This theorem is called the shoelace theorem because when you list the vertices vertically, by drawing "shoelaces" between the coordinates you can easily organize your computations.

Checkpoint 7.1. Find the area of a triangle with vertices $(1, 2)$, $(4, 6)$, and $(12, 3)$

8 Exercises

1. The endpoints of a line segment are (2, 3) and (8, 15). What is the sum of the coordinates of the midpoint of the segment?
2. What is the value of x if $-\frac{2}{3}(x-5) = \frac{3}{2}(x+1)$?
3.If $\frac{x+4}{x+3} = 4$, find the value of $\frac{1}{x+3}$. (Alcumus)
4.A quadrilateral has vertices at $(0,1)$, $(3,4)$, $(4,3)$ and $(3,0)$. Its perimeter can be expressed in the form $a\sqrt{2} + b\sqrt{10}$ with a and b integers. What is the sum of a and b? (Alcumus)
5.Points $A(3,5)$ and $B(7,10)$ are the endpoints of a diameter of a circle graphed in a coordinate plane. How many square units are in the area of the circle? (Alcumus)
6. In the overlapping triangles $\triangle ABC$ and $\triangle ABE$ sharing common side AB, $\angle EAB$ and $\angle ABC$ are right angles, $AB = 4$, $BC = 6$, $AE = 8$, and $\overline{AC}$ and $\overline{BE}$ intersect at D. What is the difference between the areas of $\triangle ADE$ and $\triangle BDC$? (2004 AMC 12A #8)
7. Right triangle ABC on the coordinate plane has A at (0,6), B at (0,0), and C at (8,0). If D and E are at the midpoints of AB and BC respectively, and if AE and CD intersect at F, what is the sum of the x and y coordinates of F?
8. What is the area enclosed by the graph of $|3x| + |4y| = 12$? (2005 AMC 12B #7)
9. A class collects 50 dollars to buy flowers for a classmate who is in the hospital. Roses cost 3 dollars each, and carnations cost 2 dollars each. No other flowers are to be used. How many different bouquets could be purchased for exactly 50 dollars? (2008 AMC 10A #8)
10. The points $(0,4)$ and $(1,3)$ lie on a circle whose center is on the x-axis. What is the radius of the circle? (Alcumus)