Chapter 2

Sequences and Series

Let's start off with a few definitions.

Definition. A *sequence* is an enumerated list of numbers.

Definition. A *series* is a sum of consecutive terms of a sequence.

Notice that a sequence can be infinite and that a series doesn't necessarily have to sum from the beginning of the sequence. Conventionally, we denote the kth term of a sequence $\{a\}$ by a_k. Now, let's look at a very important type of sequence and its corresponding series.

2.1 Arithmetic Sequences

Definition. An *arithmetic sequence* is a sequence of real numbers in which the difference between any two consecutive terms is constant. This difference is known as the *common difference*.

For example, the sequence

$$2, 4, 6, 8, 10, \ldots$$

is an example of an arithmetic sequence with common difference 2. We denote the common difference by d and the nth term of the sequence by a_n, with n taking on whole number values (so that the first term of the sequence is a_1). In mathematical terms, a sequence is an arithmetic sequence if and only if there exists a real constant d such that for all integers $n > 1$,

$$a_n = a_{n-1} + d$$

Using the definition above, we have that for any arithmetic sequence $\{a_n\}_{n\geq 1}$, the nth term satisfies

$$a_n = a_1 + (n-1)d,$$

where n is a positive integer.

Let S_n be the sum of the first n terms of an arithmetic sequence $\{a_n\}_{n\geq 1}$. We call S_n an *arithmetic series*. Then by the above observations, we have

$$S_n = a_1 + (a_1 + d) + \cdots + (a_1 + (n-1)d); \tag{2.1}$$

$$\text{and} \qquad S_n = (a_1 + (n-1)d) + (a_1 + (n-2)d) + \cdots + a_1. \tag{2.2}$$

By adding the corresponding terms in equations (2.1) and (2.2), we get

$$\begin{aligned}2S_n &= (2a_1 + (n-1)d) + (2a_1 + (n-1)d) + \cdots + (2a_1 + (n-1)d)\\ &= n(2a_1 + (n-1)d).\end{aligned}$$

Thus,

$$S_n = \frac{n(2a_1 + (n-1)d)}{2} = \frac{n(a_1 + a_n)}{2}.$$

This result should intuitively make sense; the sum of the first n terms of an arithmetic sequence is just n times the average of the first and the last terms. Another useful form of S_n, in terms of the first term, a_1, and the common difference, d, is given below:

$$S_n = na_1 + \frac{(n-1)nd}{2}.$$

To see if you have gotten the gist of arithmetic sequences, try the following checkpoint.

Checkpoint 2.1. Consider the infinite arithmetic sequence $11, 17, 23, 29, 35, \ldots$

1. What is the first term and the common difference of this sequence?
2. What is the sum of the first 10 terms in this sequence?

A common question that arises in competition math is to find the number of terms in an arithmetic sequence, as in the following example.

Example 2.1. How many terms are in the arithmetic sequence $8, 11, 14, \ldots, 35$?

Solution. We solve this problem by answering the following, more general question:

How many numbers are in the sequence $n, n+1, \ldots, m-1, m$? Equivalently, how many integers are between n and m, inclusive?

Intuitively, the answer seems to be $m - n$. However, this is **not** the case. This may seem puzzling, but as an example, let $n = 1$. How many terms are in the list $1, 2 \ldots m$? It is clearly just m, as this sequence just counts itself. But if we had used the formula of $m - n$, we would have obtained $m - 1$, which is incorrect.

Instead, the answer is $m - n + 1$. If we subtract $n - 1$ from each term in the sequence $n, n+1, \ldots, m$, we obtain the new sequence $1, 2, \ldots, m - (n-1)$. How many terms are in this sequence? It is simply $m - (n-1) = m - n + 1$. Note that by modifying each term of a sequence, we are not changing the number of terms in the sequence. This is why we were able to subtract $n - 1$ from each term while preserving the length of the sequence.

Coming back to our original problem, we see that our sequence is not in the form $n, n+1, \ldots m$. In other words, it is not made up of consecutive integers. However, we can transform this sequence into a sequence of consecutive integers.

First, we see that the common difference of this arithmetic sequence is 3. So dividing each term by 3 will give us a sequence with a common difference of 1. However, if we divide by 3, we get fractional terms instead of consecutive integers, which may be a bit too messy.

To remedy this, we notice that the integers in the sequence have a remainder of 2 when divided by 3, which motivates us to subtract 2 from each term in the sequence and subsequently divide by 3. Doing so, we get

$$8, 11, \ldots 32, 35 \quad \rightarrow \quad 6, 9, \ldots 30, 33 \quad \rightarrow \quad 2, 3, \ldots 10, 11.$$

Now, we have transformed the original sequence to a sequence of consecutive integers. Using the answer to the sub-question, we obtain that there are $11 - 2 + 1 = \boxed{10}$ integers in the sequence. △

As seen in the previous example, transforming sequences is a useful technique to count the number of terms in an arithmetic sequence. Let's try a few more examples involving arithmetic sequences.

Example 2.2. Given an arithmetic sequence with first term a_1 and common difference d, find a formula for the sum

$$a_m + a_{m+1} + \ldots + a_k$$

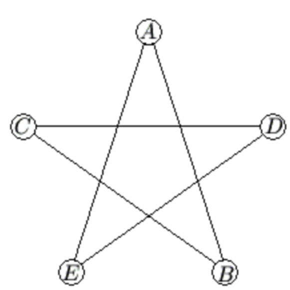
where $k \geq m$.

Solution. Notice that the sequence $a_m, a_{m+1}, \ldots a_k$ is also arithmetic, and has the same common difference as $\{a\}$! Let us define the new arithmetic sequence $\{b\}$ such that $b_1 = a_m, b_2 = a_{m+1} \ldots b_{k-m+1} = a_k$. Now, our desired sum can be computed by finding the sum of the first n terms of the arithmetic sequence $\{b\}$, which can be done using our formula for S_n. We obtain our sum to be

$$\frac{(k-m+1)(a_m + a_k)}{2}.$$

As expected, this quantity is the average of the first and last terms of our sequence multiplied by the number of terms in our sequence. △

Example 2.3. In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?
Source: AMC 10



Solution. First, let's find the sum of the arithmetic sequence. Since each vertex lies on exactly two sides of the star, the sum of the arithmetic sequence is $2(3+5+6+7+9) = 60$. Since the middle term of an arithmetic sequence is just the average of all the terms, our answer is simply $\frac{60}{5} = \boxed{12}$. △

Example 2.4. In an arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is $\frac{r^2}{s^2}$ $(r \neq s)$. Find the ratio of the 8th term to the 23rd term. *Source: ARML*

Solution. Let the first term of the arithmetic progression be a and the common difference be D. We know that the sum of the first r terms is $ra + \frac{r(r-1)}{2}d$, and that the sum of the first s terms is $sa + \frac{s(s-1)}{2}d$ (notice that we wrote the sums in terms of a and d rather than the first and last terms). This gives us the equation

$$\frac{ra + \frac{r(r-1)}{2}d}{sa + \frac{s(s-1)}{2}d} = \frac{r^2}{s^2}.$$

Dividing both sides of the equation by $\frac{r}{s}$ and then subsequently multiplying the numerator and denominator of the LHS by 2, we get

$$\frac{2a + (r-1)d}{2a + (s-1)d} = \frac{r}{s}. \tag{2.3}$$

△

Now, the problem asks us to find the ratio of the 8th term to the 23rd term, i.e.

$$\frac{a+7d}{a+22d}.$$

Notice that by substituting $r = 15$ and $s = 45$ in to (2.3), we get the desired expression on the LHS. Hence, our answer is $\frac{15}{45} = \frac{1}{3}$.

Checkpoint 2.2. Let $\{a\}$ be any arithmetic sequence. Given that $a_m + a_{m+1} + \ldots + a_n = S_p - S_q$, compute p and q in terms of m and n (where $S_k = a_1 + a_2 + \ldots + a_k$).

2.2 Geometric Sequences

Definition. A *geometric sequence* is a sequence of real numbers in which the ratio between any term and the term before it is constant. This ratio is known as the *common ratio*, which we denote by r.

The sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

is an example of a geometric sequence with common ratio $\frac{1}{2}$. A sequence $\{a\}$ is geometric if and only if there exists a real constant r such that for all integers $n > 1$,

$$\frac{a_n}{a_{n-1}} = r.$$

Using the definition above, it follows that the nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1},$$

where n is a positive integer.

The following example should give you a basic sense of how to deal with geometric sequences.

Example 2.5. A tree doubled its height every year until it reached a height of 32 feet at the end of 6 years. What was the height of the tree, in feet, at the end of 3 years? *Source: MATHCOUNTS*

Solution. Although it is not mentioned here, this problem uses a geometric sequence. Since the height of the tree doubles every year, the common ratio of the sequence of annual tree heights is 2. Here, we are given that the tree is 32 feet at 6 years, so we have to backtrack to find the height of the tree at 3 years. Since the common ratio is 2, the height of the tree at year $n-1$ is half of the height of the tree at year n. Because we have to find the height of the tree 3 years earlier, we have to multiply 32 by $\left(\frac{1}{2}\right)^3$. Doing so gives us an answer of 4 feet. △

Checkpoint 2.3. What is the 5th term of a geometric sequence that has the first term of 16 and a common ratio of $\frac{3}{2}$?

Let S_n denote the sum of the first n terms of a finite geometric sequence. We call S_n a *geometric series.* Then

$$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1} \tag{2.4}$$

and

$$S_n \cdot r = a_1 r + a_1 r^2 + \cdots + a_1 r^n. \tag{2.5}$$

By subtracting (2.5) from (2.4), we get

$$(1-r)S_n = a_1 - a_1 r^n.$$

Thus,

$$S_n = \frac{a_1 - a_1 r^n}{1-r} = a_1 \left(\frac{1-r^n}{1-r}\right).$$

From this, we've found an expression for a finite geometric series. Now, how do we sum an infinite geometric sequence, and is the sum always infinite?

For an infinite geometric sequence that converges to 0 (i.e. as n increases, a_n gets closer and closer to 0), the geometric series (i.e. the sum of the terms of the geometric sequence) is finite and can be evaluated. It turns out that an infinite geometric sequence converges to 0 if and only if $|r| < 1$. In this case, since $a_1 r^{n-1}$ goes to 0 as n increases to ∞, the geometric series is just

$$S_\infty = \frac{a_1}{1-r}.$$

Now, let's consider the case when $|r| > 1$. Notice that as n approaches ∞, $|a_n|$ also approaches ∞ (in more advanced terms, $\lim_{n\to\infty} |a_n| = \infty$). Therefore, in this case, the geometric series is infinite or undefined.

What happens when $|r| = 1$? Each of the terms will just be a constant value, so the sum again is infinite. Let's summarize the above discussion in the following theorem.

Theorem 2.1. *Let $\{a\}$ be a geometric sequence with common ratio r.*

1. *If $\{a\}$ has length n, then the sum of the terms of $\{a\}$ is*

$$a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

where $r \neq 1$. If $r = 1$, then the sum of the terms of $\{a\}$ is simply an.

2. *If $\{a\}$ has infinitely many terms and $|r| < 1$, then the sum of the terms of $\{a\}$*

$$\frac{a_1}{1 - r}.$$

3. *If $\{a\}$ has infinitely many terms and $r \geq 1$, then the sum of the terms of $\{a\}$ is $\pm\infty$ (which has the same sign as a_1).*

4. *If $\{a\}$ has infinitely many terms and $r \leq -1$, then the sum of the terms of $\{a\}$ is undefined.*

To get a grasp of the previous theorem, let's try an example.

Example 2.6. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. Find the common ratio of the original series. *Source: AIME*

Solution. Let $\{a\}$ be the infinite geometric sequence and let its common ratio be r. From the problem statement, we see that the series of $\{a\}$ must be finite; hence, the series is equal to

$$\frac{a_1}{1 - r}.$$

Now, let $\{b\}$ be the sequence obtained by squaring each term of $\{a\}$. Notice that this new sequence is in fact geometric with common ratio of be is r^2. Thus, the series of $\{b\}$ is

$$\frac{a_1^2}{1 - r^2}.$$

This gives us the equation

$$\frac{a_1^2}{1 - r^2} = 10 \frac{a_1}{1 - r}.$$

Since $a_1, 1 - r \neq 0$, we can divide both sides by $\frac{a_1}{1-r}$ to get

$$\frac{a_1}{1 + r} = 10. \tag{2.6}$$

We also know that

$$\frac{a_1}{1-r} = 2005. \tag{2.7}$$

Dividing equation (2.7) by (2.6), we get

$$\frac{1+r}{1-r} = \frac{401}{2}.$$

Multiplying both sides by $1-r$ gives us

$$1+r = \frac{401}{2} - \frac{401}{2}r.$$

Hence,

$$r = \frac{\frac{401}{2}-1}{\frac{401}{2}+1} = \boxed{\frac{399}{403}}.$$

△

Checkpoint 2.4. The sum of an infinite geometric series is 27 times the series that results if the first three terms of the original series are removed. What is the value of the series' common ratio? *Source: AoPS*

2.3 Arithmetico-Geometric Sequences

Definition. An *arithmetico-geometric sequence* is a sequence in which each term is the product of a term from an arithmetic sequence and a term from a geometric sequence.

We will now find the sum of a general, infinite arithmetico-geometric sequence,

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \cdots$$

where the a_i form an arithmetic sequence (with common difference d) and the b_i form a geometric sequence (with common ratio $r > 1$).

$$S = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots$$

$$rS = \frac{a_1 r}{b_1} + \frac{a_2 r}{b_2} + \frac{a_3 r}{b_3} + \cdots$$

Notice that $\frac{r}{b_k} = \frac{r}{rb_{k-1}} = \frac{1}{b_{k-1}}$. Hence,

$$rS = \frac{a_1 r}{b_1} + \frac{a_2}{b_1} + \frac{a_3}{b_2} + \cdots$$

and from above,

$$S = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots$$

So by matching terms with the same denominator, we get

$$rS - S = \frac{a_1 r}{b_1} + \frac{a_2 - a_1}{b_1} + \frac{a_3 - a_2}{b_2} + \frac{a_4 - a_3}{b_3} + \cdots$$

But by the definition of arithmetic sequences, we have $a_k - a_{k-1} = d$, so

$$rS - S = \frac{a_1 r}{b_1} + \frac{d}{b_1} + \frac{d}{b_2} + \frac{d}{b_3}.$$

Note that $\frac{d}{b_1}, \frac{d}{b_2}, \frac{d}{b_3} \dots$ is a geometric sequence, with common ration $\frac{1}{r}$. Using our formula for the infinite sum of a geometric sequence, we have

$$(r-1)S = \frac{a_1 r}{b_1} + \frac{\frac{d}{b_1}}{1 - \frac{1}{r}}.$$

Hence,

$$S = \frac{a_1 r + \frac{d}{1-\frac{1}{r}}}{b_1(r-1)}. \tag{2.8}$$

We leave it to the reader to make this expression for S nicer. We can use a similar strategy to get the sum of a finite arithmetico-geometric sequence. The important technique to glean from this is multiplying an arithmetico-geometric series by the common ratio of the denominator, and taking the difference of that with the original series to observe nice simplifications.

Let's try a few examples cement this technique.

Example 2.7. Find

$$\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots.$$

Solution. Let $S = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$. We can multiply S by 2 and subtract S from that to get a cancellation of terms.

$$2S = \frac{1}{2^0} + \frac{2}{2^1} + \frac{3}{2^2} + \cdots$$
$$S = 2S - S = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots.$$

Using the formula for the sum of an infinite geometric series for the above expression, we get

$$S = \frac{\frac{1}{2^0}}{1 - \frac{1}{2}} = \boxed{2}.$$

We can also use the formula outlined in equation (2.8) with $a_1 = 1$, $b_1 = 2$, $d = 1$, and $r = 2$ to get the same answer. △

Example 2.8. Evaluate

$$1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + 100 \cdot 2^{99}.$$

Source: Brilliant

Solution. Let S be the desired sum. Then

$$2S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + 100 \cdot 2^{100}$$
$$S = 2S - S = -(1 + 2 + 2^2 + 2^3 + \cdots + 2^{99}) + 100 \cdot 2^{100}.$$

Using the formula for the sum of a geometric sequence, we have

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{99} = 2^{100} - 1.$$

Hence, our answer is

$$100 \cdot 2^{100} - (2^{100} - 1) = \boxed{99 \cdot 2^{100} + 1}.$$

△

Example 2.9. Evaluate

$$\sum_{k=1}^{\infty} \frac{k^2}{3^k}.$$

Solution. Let S be the desired sum. Then

$$3S = 1^2 + \frac{2^2}{3} + \frac{3^2}{3^2} + \frac{4^2}{3^3} + \cdots$$
$$3S - S = 2S = 1 + \frac{3}{3} + \frac{5}{3^2} + \frac{7}{3^2} + \cdots.$$

Now, notice that the LHS (besides the first term) is an infinite arithmetico-geometric series. Using (2.8) by setting $a_1 = 3$, $b_1 = 3$, $d = 2$, and $r = 3$, we get that

$$\frac{3}{3} + \frac{5}{3^2} + \frac{7}{3^2} + \cdots = 2.$$

Hence, $2S = 3$, giving us $S = \boxed{\frac{3}{2}}$. △

In the previous example, it turns out we had a series called a *quadratic arithmetico-geometric series*. This is because after looking at $rS - S$, we reduced the problem into solving the common linear arithmetico-geometric series. In general, by applying the $rS - S$ technique, we can reduce a high-degree arithmetico-geometric series to a simple, linear one, which we can easily solve.

Checkpoint 2.5. Find the value of S given $S = \frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \ldots + \frac{2k-1}{3^k} + \ldots$

2.4 Telescopic Sums and Products

When trying to find the sum or product of a sequence of terms, a useful technique is to write each term in such a way that when you rewrite all terms, almost all of them cancel in some way. This technique is called *telescoping*, and can often help simplify sums and products of terms, especially ones that contain fractions, trigonometric functions, or factorials. The general idea behind telescoping is given in the following theorem.

Theorem 2.2. *Given any sequence $\{a\}$, we have*

$$\sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$$

Proof. To see this, simply expand out the solution. All terms cancel except the a_{n+1} and $-a_1$. □

Let's introduce the technique with a few examples.

Example 2.10. Find

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}.$$

Solution. First, let's look at the kth term of this sum, which is $\frac{1}{k(k+1)}$. Notice that we have the relationship

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}.$$

Substituting this identity into the sum and canceling like terms, we get

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\\ &\quad+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &=\frac{1}{1}-\frac{1}{n+1}\\ &=\boxed{\frac{n}{n+1}}.\end{aligned}$$

In this particular problem, we dealt with a denominator of $k(k+1)$. A more general version would be having a denominator of $k(k+c)$ for some integer c. In this case, we use the identity

$$\frac{1}{k(k+c)}=\frac{1}{c}\left(\frac{1}{k}-\frac{1}{k+c}\right)$$

△

Example 2.11. Evaluate

$$\sum_{k=1}^{n} k!(k^2+1).$$

Solution. In general, when we want a sum to telescope, we aim to write the summand as a difference of terms which would lead to a cancellation when summing those terms. Motivated by this, we do the following manipulations:

$$\begin{aligned}k!(k^2+1) &= k^2\cdot k!+k!\\ &= k(k\cdot k!)+k!\\ &= k((k+1)!-k!)+k!\\ &= (k+1)!k-k!(k-1).\end{aligned}$$

This looks good, because if we let $f(m)=(m+1)!m$, then $f(m-1)=m!(m-1)$. Now, we evaluate the desired sum.

$$\begin{aligned}\sum_{k=1}^{n} k!(k^2+1) &= \sum_{k=1}^{n}(k+1)!k - k!(k-1)\\ &= \sum_{k=1}^{n} f(k) - f(k-1)\\ &= f(1) - f(0) + f(2) - f(1) + \ldots + f(n-1) - f(n-2) + f(n) - f(n-1)\\ &= f(n) - f(0)\\ &= (n+1)!n - 1!\cdot 0\\ &= \boxed{n\cdot(n+1)!}\end{aligned}$$

△

If a problem asks to find the sum of a very long sequence that isn't arithmetic, geometric, or arithmetico-geometric, it most likely uses telescoping. These sums can be hard to crack, so the best way to find the solution is to try smaller numbers and notice a pattern. After doing that, then prove that such pattern is correct, which should verify that your final answer is correct.

Checkpoint 2.6. For a positive integer n, let

$$f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots .$$

Find

$$\sum_{n=2}^{\infty} f(n).$$

Source: AoPS

2.5 Problems

1. ⋆ Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X? *Source: AMC 8*

1				25
		X		
17				81

2. ⋆ BoatsRUs built 7 canoes in January of this year and then each subsequent calendar month they built twice the number of canoes they had built the previous month. How many total canoes were built by BoatsRUs by the end of May of this year? *Source: AoPS* Ans: 217

3. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain? *Source: AMC 12*

4. ⋆ To place the first paving stone in a path, Alex starts at the crate of stones, walks three feet, places the stone, and returns to the crate. For each subsequent stone, Alex walks two feet farther each way. Alex will place the first 50 stones in a path. After returning to the crate from placing the 50^{th} stone, what is the total distance Alex walked, in feet? *Source: AoPS*

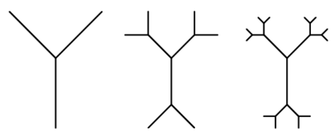


Figure 1 Figure 2 Figure 3

5. ⋆ If you continue this pattern in which each line-segment extremity is replaced by a gradually smaller Y in the next figure, in the manner shown, how many endpoints will Figure 5 have?

6. ⋆ An airplane climbs 100 feet during the first second after takeoff. In each succeeding second it climbs 100 feet more than it climbed during the previous second. How many seconds does it take for the plane to reach an altitude of 12,000 feet above its takeoff height? *Source: MATHCOUNTS*

7. ⋆ When reading a book, Charlie made a list by writing down the page number of the last page he finished reading at the end of each day. (He always finished reading a page that he started.) His mom thought his list indicated the amount of pages he had read on each day. At the end of the 8^{th} day of reading, she added the numbers on his list and thought Charlie had read 432 pages. If Charlie started reading the book on page one, and he read the same amount of pages each day of this eight- day period, how many pages did he actually read by the end of the 8^{th} day? *Source: MATHCOUNTS*

8. ⋆ Calculate $\frac{1}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{4}{7} \cdots \frac{49}{52} \cdot \frac{50}{53}$. *Source: MATHCOUNTS*

9. ⋆ What is the sum of all of the multiples of 3 between 100 and 200? *Source: MATHCOUNTS*

10. ⋆ The nth term of a sequence is $a_n = (-1)^{n+1}(3n+2)$. What is the value of $a_1 + a_2 + \cdots + a_{100}$? *Source: MATHCOUNTS*

11. ⋆ Compute
$$\sum_{n=1}^{\infty} \frac{3n-1}{2^n}.$$
Source: AoPS

12. ⋆ Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages. *Source: ARML*

13. ⋆⋆ How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression? *Source: AMC 10*

14. ⋆⋆ An arithmetic sequence has first term a and common difference d. If the sum of the first ten terms is half the sum of the next ten terms, what is the ratio $\frac{a}{d}$? *Source: AoPS*

15. ⋆⋆ Compute the product

$$\frac{(1998^2 - 1996^2)(1998^2 - 1995^2)\cdots(1998^2 - 0^2)}{(1997^2 - 1996^2)(1997^2 - 1995^2)\cdots(1997^2 - 0^2)}.$$

Source: AoPS

16. ⋆⋆ For each positive integer k, let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, 10, \ldots$. For how many values of k does S_k contain the term 2005? *Source: AIME*

17. ⋆⋆ It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$. *Source: AIME*

18. ⋆⋆ Prove that if there is a perfect square in an infinite arithmetic sequence of positive integers, then there must be infinitely many perfect squares in that sequence. *Source: AoPS*

19. ⋆⋆ One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. Find the ratio of the total area of the green regions to the area of the circle of radius 100. *Source: AIME*

20. ⋆⋆ The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence. *Source: AIME*

21. ⋆⋆ A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression? *Source: AMC 12*

22. ⋆⋆ The geometric series $a + ar + ar^2 \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$? *Source: AMC 12*

23. ⋆⋆ Evaluate the product

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{9^2}\right)\left(1 - \frac{1}{10^2}\right)$$

Source: AHSME

24. $\star\star$ Evaluate the sum

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}.$$

25. $\star\star$ For $-1<r<1$, let $S(r)$ denote the sum of the geometric series

$$12+12r+12r^2+12r^3+\cdots.$$

Let a between -1 and 1 satisfy $S(a)S(-a)=2016$. Find $S(a)+S(-a)$. *Source: AIME*

26. $\star\star$ Let $a+ar_1+ar_1^2+ar_1^3+\cdots$ and $a+ar_2+ar_2^2+ar_2^3+\cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is r_1+r_2? *Source: AMC 12*

27. $\star\star$ In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms. *Source: AIME*

28. $\star\star$ If $T_n=1+2+3+\cdots+n$ and

$$P_n=\frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdot\cdots\cdot\frac{T_n}{T_n-1}$$

for $n=2,3,4,\cdots$, find the integer closest to P_{1991}. *Source: AHSME*

29. $\star\star$ A sequence of numbers $x_1,x_2,x_3,\ldots,x_{100}$ has the property that, for every integer k between 1 and 100, inclusive, the number x_k is k less than the sum of the other 99 numbers. Find x_{50}. *Source: AIME*

30. $\star\star$ Evaluate

$$\sum_{k=1}^{n} k!(k^2+1).$$

31. $\star\star$ (Gabriel's staircase) Show that

$$\sum_{k=1}^{\infty} kr^k=\frac{r}{(1-r)^2},\quad \text{for } 0<r<1$$

32. Find the eighth term of the sequence 1440, 1716, 1848, $\ldots$, whose terms are formed by multiplying the corresponding terms of two arithmetic sequences. *Source: AIME*

33. $\star\star\star$ Find

$$\sum_{n=2}^{\infty}\frac{n^3-1}{n^3+1}.$$

34. $\star\star\star$ Let $x=\frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$ *Source: AIME*

35. $\star\star\star$ Find

$$\sum_{k=1}^{n}\frac{k}{k^4+k^2+1}.$$

Source: MATH LEAGUE

36. ⋆⋆⋆ Say that a positive integer n is smooth if $\frac{1}{n}$ has a terminating decimal expansion. (Note that 1 is smooth.) Compute the value of the infinite series

$$\sum_n \frac{1}{n^3},$$

where n ranges over all smooth positive integers. *Source: Math Prize For Girls*

37. ⋆⋆⋆ The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$\sum_{n=0}^{7} \log_3(x_n) = 308$ and $56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57$,

find $\log_3(x_{14})$. *Source: AIME*

38. ⋆⋆⋆ Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer. *Source: AIME*

39. ⋆⋆⋆ The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k. *Source: AIME*

40. ⋆⋆⋆ Find the least positive integer n such that

$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}$. *Source: AIME*

41. ⋆⋆⋆ The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}. *Source: AIME*

42. ⋆⋆⋆ A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms $a_{2n-1}, a_{2n}, a_{2n+1}$ are in geometric progression, and the terms a_{2n}, a_{2n+1}, and a_{2n+2} are in arithmetic progression. Let a_n be the greatest term in this sequence that is less than 1000. Find $n + a_n$. *Source: AIME*