Iowa City Math Circle Handouts
Recursion

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1 Recursive Sequences

A *recursive* sequence, also known as a *recurrence relation*, is a sequence in which the n-th term is defined in terms of the first $n-1$ terms of the sequence and n (usually, the definition does not involve all of the first $n-1$ terms). Put simpler, a recursive sequence is defined in terms of itself. For example, the Fibonacci sequence is a recursive sequence and can be defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$, where n is any integer greater than 1. We then see that

$$\begin{aligned} F_2 &= 1 + 0 = 1 \\ F_3 &= 1 + 1 = 2 \\ F_4 &= 2 + 1 = 3 \\ F_5 &= 3 + 2 = 5, \end{aligned}$$

and so on. When defining a recursive function, you may notice that

Checkpoint 1.1. Find the 5th term ($n = 5$) in the following recursive sequence: $a_1 = 3, a_n = 2a_{n-1} + 3$.

In a later section, we will discuss a general method for finding a closed-form expression (i.e. one that doesn't involve other terms of the sequence) for the n-th term of some recursive sequences. However, For some types of recurrence relations, we may be able to find a closed-form for the n-th term more easily (by noticing a pattern). Consider the recurrence defined by $a_0 = 1$ and $a_n = n \cdot a_{n-1}$ for $n \in \mathbb{Z}^+$. We see that

$$\begin{aligned} a_n &= n \cdot a_{n-1} \\ &= n \cdot (n-1) \cdot a_{n-2} \\ &= n \cdot (n-1) \cdot (n-2) \cdot a_{n-3} \\ &= n \cdot (n-1) \cdots 1 \cdot a_0 \\ &= n!. \end{aligned}$$

From the factorial recurrence relation and other examples we will see in combinatorics, recursion helps us solve problems faster, computationally and practically.

Checkpoint 1.2. Find a closed-form expression for the n-th term of the recursive sequence defined by $a_1 = 2$ and $a_n = n(n-1) \cdot a_{n-1}$

In summary, the main objective of recursion is to represent the solution to the current case in terms of the solutions of smaller cases (similar to the method of induction!). In later sections, we'll see how we can use this principle idea of recursion to solve harder problems in other topics.

An explicit representation, or a closed-form, means that we can directly plug in n to find the value of $f(n)$ without needing to know any of the previous values of f. Finding a closed form can be a complicated process, so let's take a look at some basic examples first.

Example 1.1. Find a closed form for the sequence $a_1 = 4$, $a_n = a_{n-1} - 5$.

Solution. Let's take a look at what this sequence does. When we evaluate the first few terms of this sequence, we get $4, -1, -6, -11, -16$, etc. This is clearly an arithmetic sequence, with first term 4 and common difference -5. We can now use the arithmetic sequence equation to get a closed form: $a_n = 4 + -5(n-1)$ or $a_n = 9 - 5n$. Note that we had an $n - 1$ term instead of n in the common difference because the first term of our sequence is at $n = 1$, not $n = 0$. △

This is one of the simpler equations to find a closed form because it doesn't involve many complicated equations. Try this similar problem with the same strategy, by starting with listing the first few terms of the sequence.

Checkpoint 1.3. Find a closed form for the sequence $a_1 = 7$, $a_n = 5a_{n-1}$.

There are many sequences that involve complicated expressions when finding a closed form. Those will be discussed in the next chapter.

2 Solving Cyclic Recursions

Sometimes a problem involves a *periodic* (or cyclic) recurrence. This means that the values produced by the recursive relation will repeat regularly. In cases like this, solving the recursive equation is usually not necessary. Instead, we can find the pattern and use it to obtain the answer.

Example 2.1. Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Find t_{2020}. *Source: AIME*

Solution. The first thing to note is that the question is asking for the 2020th term. This means that trying to evaluate it directly would simply just take too long, so let's try to look at the problem in a different way. It is likely that this problem has a cycle, because if not then the problem's answer would likely be too complicated to compute without a calculator. Let's try evaluating the first few terms in this sequence.
To simplify things, let $a = t_1$ and $b = t_2$. Then, $t_3 = \frac{5b+1}{25a}$, $t_4 = \frac{5a+5b+1}{125ab}$, $t_5 = \frac{5a+1}{25b}$, $t_6 = a$, and $t_7 = b$. We can then see quickly that $t_6 = t_1$ and $t_2 = t_7$. Since this recursive sequence only uses the last two terms, we know that this sequence has a cycle of 5. Now, let's take a look at what the problem asks for. It asks for the 2020th term, but since the sequence has a cycle of 5, we really only need to find the 5th term (2020 and 5 are both multiples of 5). It now suffices to evaluate t_5: $t_5 = \frac{5a+1}{25b} = \boxed{\frac{101}{525}}$. △

Checkpoint 2.1. Let a_n be the remainder when F_n is divided by 3 (equivalently, $a_n = F_n \mod 3$), where F_n is the Fibonacci sequence, defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$. Determine the value of a_{2020}.

3 Solving Linear Homogeneous Recursions

A *linear homogeneous recurrence relation* is a recurrence relation defined by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m},$$

for all $n \geq k$, where the c_i's are real constants and m and k are non-negative integers with $k > m$ and $c_m \neq 0$.

In other words, a linear homogeneous recurrence is one that is linear, homogeneous, and has constant coefficients. This sequence is linear since all the a_i's on the right-hand side are raised to the power one (not squared, cubed, etc.). This relation is also homogeneous since every term on the right-hand side all have the same degree which is 1.(Note that a constant has a power of 0!) If you are not familiar with the phrase "degree of a term", it simply refers to the sum of the exponents of the variables in the term. To see if you have a grasp of this definition, try the following example.

Example 3.1. Which of the following recurrence relations are linear homogeneous recurrence relations? For the ones that are not, which of the three criteria (linearity, homogeneity, and constant coefficients) do they not meet?

1. $a_n = na_{n-1}$
2. $a_n = a_{n-1}a_{n-2}$
3. $a_n = ca_{n-1}$ for some constant c.
4. $a_n = 2a_{n-1} + a_{n-2} + 3$
5. $a_n = a_{n-1}^2 + a_{n-2}^2$

6. $a_n = 2a_{n+1} - 3a_{n-1}$
7. $a_n = 3a_{n-1}^2 + 2a_{n-2}$
8. $a_n = 2a_{n-1} - 3a_{n-2}$

Solution. Only items 3, 6, and 8 are linear homogeneous recurrence relations.

1. This relation is linear and homogeneous, but does not have constant coefficients since the coefficient of a_{n-1}, namely n, is not a constant.
2. This relation is not linear (it's quadratic since the term $a_{n-1}a_{n-2}$ has degree 2) and doesn't have constant coefficients. It is homogeneous, though, as it's right-hand side contains one term.
3. This relation is linear, homogeneous (all the terms have degree 1), and has constant coefficients (namely 2 and 3).
4. This relation is linear and has constant coefficients, but is not homogeneous since the last term is constant and has degree 0 (whereas the other terms have degree 1).
5. This relation is not linear but is homogeneous since each of its terms have degree 2. However, it has constant coefficients.
6. We can rearrange this relation to get

   $$a_{n+1} = .5a_n + 1.5a_{n-1},$$

   which is linear, homogeneous, and has constant coefficients.
7. This relation is not linear nor homogeneous since its two terms have degrees 2 and 1, respectively. It has constant coefficients, though.
8. This relation is linear, homogeneous (all terms have degree 1), and has constant coefficients.

△

Now, we'll introduce another definition. The *characteristic polynomial* of the linear homogeneous recurrence

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ma_{n-m}$$

is the polynomial

$$p(x) = x^m - c_1x^{m-1} - c_2x^{m-2} - \cdots - c_{m-1}x - c_m.$$

Now, let's state a theorem that gives us a solution to any linear homogeneous recurrence whose characteristic polynomial has distinct roots.

Theorem 3.1. *Suppose we have the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m}$$

with characteristic polynomial

$$p(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1} x - c_m.$$

Then if $p(x)$ has distinct roots $r_1, r_2, \ldots, r_m$, any sequence $\{a_n\}$ satisfies this recurrence if and only if

$$a_k = b_1 r_1^k + b_2 r_2^k + \cdots + b_m r_m^k,$$

for all $k = 0, 1, \ldots, n$, where $b_1, b_2, \ldots, b_m$ are constants.

You may wonder how the above theorem gives us a closed form, considering the b_i's are unknown constants. We can get around this by plugging in m values in the sequence $\{a_n\}$ to get an $m \times m$ system of linear equations, which can be solved without much trouble. Let's understand this theorem with an example.

Example 3.2. Find a closed-form expression for the n-th term of the Fibonacci sequence.

Solution. The Fibonacci sequence is given by the recurrence $F_n = F_{n-1} + F_{n-1}$. As this is clearly a linear homogeneous relation, we see that its characteristic polynomial is $p(x) = x^2 - x - 1$. Using the quadratic formula, the roots of the polynomial are $\frac{1 \pm \sqrt{5}}{2}$. Applying the previous theorem, we get that the n-th term of the Fibonacci sequence is given by

$$F_n = b_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + b_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n,$$

for some constants b_1 and b_2. But how do we find these constants? We know that $F_0 = 0$ and $F_1 = 1$. Plugging this in, we have that

$$\begin{cases} b_1 + b_2 & = 0 \\ b_1 \left(\frac{1+\sqrt{5}}{2} \right) + b_2 \left(\frac{1-\sqrt{5}}{2} \right) & = 1. \end{cases}$$

Plugging the first equation into the second, we get that

$$b_1 \left(\frac{1 + \sqrt{5}}{2} \right) - b_1 \left(\frac{1 - \sqrt{5}}{2} \right) = 1.$$

Simplification yields $\sqrt{5} b_1 = 1$, or $b_1 = \frac{1}{\sqrt{5}}$. This also gives us $b_2 = -\frac{1}{\sqrt{5}}$ from the first equation. Hence, the following is the closed form for the Fibonacci sequence (also called Binet's Formula):

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

△

Checkpoint 3.1. Defined below is the Pell sequence, $\{P_n\}$, as follows:

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{if } n > 1. \end{cases}$$

Find a closed form for P_n.

Let's now take a look at a harder example, in which we show how to convert nonlinear recurrence relations to linear homogeneous relations using the logarithmic function.

Example 3.3. The sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer? *Source: AMC*

Solution. Let's examine the sequence $\{a_n\}$. It isn't linear (infact it is cubic) nor does it have constant coefficients. Therefore, this sequence is not linear homogeneous. Can we however transform this sequence into a linear homogenous sequence? A common problem solving technique to reduce the degree of a term is to take logs. Using the basic properties of logs will transforms the original nonlinear sequence to a linear sequence. Before, we apply the technique to this problem, we need to figure out what base we will be taking the log with respect to. The specific choice of the base doesn't affect our solution much in the long run, but the choice of the base of 2 makes computation a tad easier since $a_1 = \sqrt[19]{2}$. Choosing our base to be 2, we can take the log of both sides our recursion to get

$$\begin{aligned} \log_2 a_n &= \log_2 \left(a_{n-1}a_{n-2}^2\right) \\ &= \log_2 a_{n-1} + \log_2 a_{n-2}^2 \\ &= \log_2 a_{n-1} + 2\log_2 a_{n-2}. \end{aligned}$$

Now, let's define the sequence $b_n = 19\log_2 a_n$. You may be wondering we we multiplied by 19, and that's because it gives us $b_1 = 1$, so that combined with $b_0 = 0$, our sequence $\{b_n\}$ is a non-negative integer sequence. Then we have the recursion

$$b_n = b_{n-1} + 2b_{n-2}.$$

This is a linear homogenous recurrence with characteristic polynomial

$$p(x) = x^2 - x - 2.$$

The roots of p are just -1 and 2 by simple factoring or the quadratic formula. Since these roots are distinct, we can apply the main theorem from this section to get that

$$b_n = c_1(-1)^n + c_2 2^n,$$

for constants c_1 and c_2. Plugging in the values $b_0 = 0$ and $b_1 = 1$, we have the system of equations

$$\begin{cases} c_1 + c_2 & = 0 \\ -c_1 + 2c_2 = 1. \end{cases}$$

Adding the two equations and dividing by three gives us $c_2 = \frac{1}{3}$. Plugging this into the first equation gives us $c_1 = -\frac{1}{3}$. Hence,

$$b_n = -\frac{1}{3}(-1)^n + \frac{1}{3}2^n = \frac{2^n - (-1)^n}{3}.$$

Now, the problem is asking us to find (or at least consider the value of) $a_1 a_2 \cdots a_k$. We can rearrange the equation $b_n = 19 \log_2 a_n$ to get $a_n = 2^{\frac{b_n}{19}}$. Thus,

$$\begin{aligned} a_1 a_2 \cdots a_k &= 2^{\frac{b_1}{19}} 2^{\frac{b_2}{19}} \cdots 2^{\frac{b_k}{19}} \\ &= 2^{\frac{b_1 + b_2 + \cdots + b_k}{19}}. \end{aligned}$$

From our closed form for the sequence b_n, we can compute $b_1 + b_2 + \cdots + b_n$ for two cases: if n is even or if n is odd. If n is even, we get that

$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= \frac{2 - (-1) + 2^2 - 1 + 2^3 - (-1) + \cdots + 2^n - 1}{3} \\ &= \frac{2 + 2^2 + \cdots + 2^n}{3} \\ &= \frac{2^{n+1} - 2}{3}, \end{aligned}$$

using the geometric series formula. For n odd,

$$\begin{aligned} b_1 + b_2 + \cdots + b_n &= \frac{2 - (-1) + 2^2 - 1 + 2^3 - (-1) + \cdots + 2^n - (-1)}{3} \\ &= \frac{2 + 2^2 + \cdots + 2^n + 1}{3} \\ &= \frac{2^{n+1} - 1}{3}. \end{aligned}$$

Thus, to have $a_1 a_2 \cdots a_k$ being an integer, $b_1 + b_2 + \cdots + b_n$ must be divisible by 19, or $2^{n+1} - 2$ is divisible by 19 in the even case or $2^{n+1} - 1$ is divisible by 19 when n is odd. Using Fermat's Little Theorem (a special case of Euler's Theorem) we have that $2^{18} \equiv 1 \pmod{19}$. Therefore, $2^{19} - 2 \equiv 2 - 2 \equiv 0 \pmod{19}$ and $2^{18} - 1 \equiv 1 - 1 \equiv 0 \pmod{19}$. These two statements give rise to the solutions $n = 17$ and $n = 18$. These two values are the least odd and even solutions, respectively. Hence, our answer is 17. △

4 Applications in Combinatorics

Recursion can be a powerful technique to represent combinatorics problems in a simpler fashion and make solving them more straightforward. To apply recursion to such problems, we will first have to find the base case that is easy to compute. Then, we can find the recursive step, which is the relationship between the solution to the k-th case and the solutions to previous cases. This bottom-up nature of recursion is what makes is quite efficient.

Example 4.1. Find the number of 10 digit positive binary numbers that do not have a pair of consecutive 0s.

Solution. We can tackle this problem with casework, but that can get messy very quickly and is prone to error. Let's try recursion here, since it seems like we can start with smaller cases.

The base case here is simply just one digit, and with that we have one number: 1. We can then see that for two digits we have 2 numbers: 10 and 11. Now, let's take a look at the recursive step. We would have to divide this into two cases: whether the number ends with a 0 or a 1. If the number ends in 1, then we have 2 possible ways to add an extra digit (0 or 1) to the end, but if the number ends in 0, then we only have one way to add the digit since we can't append a 0 to that binary number.

With all of that in mind, we can now set up our recurrence. Let a_n be the amount of binary numbers with n digits that end in 0, and let b_n be the amount of numbers with n digits that end in 1. We find that a_1 and b_1 are both 1. Now, let's write the recursive step that we discussed above into equations:

$$a_n = b_{n-1}$$
$$b_n = a_{n-1} + b_{n-1}$$

We can use these equations to compute the final answer, as shown in the table.

n	a_n	b_n
1	0	1
2	1	1
3	1	2
4	2	3
5	3	5
6	5	8
7	8	13
8	13	21
9	21	34
10	34	55

Since $a_{10} = 34$ and $b_{10} = 55$ from the above table, we have a total of $34 + 55 = \boxed{89}$ binary numbers.

Remark: Do you see something special about the sequences a_n and b_n from the table? Why do you think that is the case? △

In this problem, we showed that recursion can save us some time when counting directly seems to be difficult. Recursion is best used when there is a clear base case and there are not a lot of steps to the case that the question asks (although this can be ignored if the recursion is simple, cyclic, or linear homogeneous). The trickiest part in recursion is finding the recursive relationship between cases. If you can figure that out, solving the rest of the problem will be relatively easy.

Next, let's take a look at a trickier problem.

Example 4.2. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14? *Source: AIME*

Solution. Like the previous problem, we can solve this by direct counting. However, doing so is prone to mistakes because of the complexity of the problem. So, let's try recursion again.

The base cases are pretty simple - there is only one such sequence of length one (B) and one sequence of length two (AA). Let's now focus on the recursive case. Like the other problem, let's denote a_n to be the number of sequences with length n that end with A, and let b_n be the number of length n sequences that end in B.

Let's focus on a_n first. We can generate a sequence with length n that ends with A by appending a string of As, with even length, to a sequence that ends with B. For example, we can generate a sequence of length 9 by appending AA to a sequence with length 7 that ends with B. Turning this into a equation yields us

$$a_n = b_{n-2} + b_{n-4} + b_{n-6} + \cdots + b_0 \text{ or } b_1.$$

We can use a similar logic for b_n, but instead of adding a string of Bs to a sequence that has a length with the same even-odd parity, we need to add it to a sequence with a length of the opposite parity. We can now write an equation for b_n:

$$b_n = a_{n-1} + a_{n-3} + a_{n-5} + \cdots + a_0 \text{ or } a_1.$$

Now that we have both equations, we can now evaluate a_{14} and b_{14} using the following table.

n	a_n	b_n
0	1	1
1	0	1
2	1	0
3	1	2
4	1	1
5	3	3
6	2	4
7	6	5
8	6	10
9	11	11
10	16	21
11	22	27
12	37	43
13	49	64
14	80	92

We can now add up a_{14} and b_{14}, or 80 and 92, to get our final answer of $\boxed{172}$.

△

Checkpoint 4.1. Ben is jumping up a flight of stairs. He can take one step or two steps in a jump. Find the number of ways he can climb up n steps, if $n = 10$.

5 Applications in Functional Equations

We can also use clever algebra manipulations to solve equations involving recursive functions. Recursive functions can just be treated as a recursive sequence, as the only main difference is notation.

Example 5.1. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$? *Source: AMC*

Solution. We can substitute $n-1$ into the equation to get

$$f(n-1) = f(n-2) - f(n-3) + n - 1.$$

We can plug this back into the original equation to get $f(n) = -f(n-3) + 2n - 1$, or equivalently, $f(n) + f(n-3) = 2n - 1$. This means that

$$\begin{aligned} f(2018) + f(2015) &= 2 \cdot 2018 - 1 \\ f(2015) + f(2012) &= 2 \cdot 2015 - 1 \\ &\vdots \\ f(5) + f(2) &= 2 \cdot 5 - 1 \end{aligned}$$

We can then compute

$$f(2018)+f(2015)-(f(2015)+f(2012))+f(2012)+f(2009)-\cdots-(f(5)+f(2)) = f(2018)-f(2)$$

which equals

$$2\cdot 2018 - 2\cdot 2015 + 2\cdot 2012 - \cdots - 2\cdot 5 = 2\cdot 3\cdot 671 = 2016.$$

Therefore $f(2018) - f(2) = 2016 \implies f(2018) = 2016 + 1 = \boxed{2017}$. △

6 Exercises

1. ⋆ Find the 5th term of the following recursive sequence: $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1}a_{n-2}$.

2. ⋆ Find the closed form for the recursive sequence: $a_1 = 10, a_n = 3a_{n-1} - 2$.

3. ⋆ Find the number of subsets of $\{1, 2, \cdots, n\}$ that contain no two consecutive elements. *Source: 112 Combinatorics*

4. ⋆ How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s? *Source: AMC*

5. ⋆ Consider the triangular array of numbers with 0, 1, 2, 3, ... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & 3 & & 4 & & 4 & & 3 & & \\
 & 4 & & 7 & & 8 & & 7 & & 4 & \\
5 & & 11 & & 15 & & 15 & & 11 & & 5
\end{array}$$

   Find the sum of the numbers in the 100th row. Your answer should use exponential notation, in simplest form. *Source: AoPS*

6. ⋆ Consider the sequence defined recursively by $u_1 = a > 0$ and

$$u_{n+1} = -\frac{1}{u_n + 1}$$

   for $n \geq 1$. Express u_{16} in terms of a. *Source: AoPS*

7. ⋆ Given $a_0 = 1$, $a_1 = 3$, and the general relation $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for $n \geq 1$, find a_3. *Source: AHSME*

8. $\star$ Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending \$1024. When she comes out of the last shop she has no money left. What is the minimum possible value of d? *Source: HMMT*

9. $\star$ For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008th term of the sequence? *Source: AMC*

10. $\star$ The sequence of numbers $t_1, t_2, t_3, \ldots$ is defined by $t_1 = 2$ and

$$t_{n+1} = \frac{t_n - 1}{t_n + 1}$$

for every positive integer n. Determine the value of t_{999}. *Source: COMC*

11. $\star$ A parking lot in a workplace has 8 slots. There are smaller cars that only take up 1 slot, and larger vans that take up two slots. There are three possible types of cars and two types of larger vans. How many possible ways can the parking lot be arranged if it is full?

12. $\star\star$ In terms of n, how many regions can we make in a plane of n lines? (For example, we can make 4 regions with 2 lines in a plane)

13. $\star\star$ A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Find a_{2019}. *Source: AMC*

14. $\star\star$ How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s? *Source: AMC*

15. $\star\star$ In the sequence 2001, 2002, 2003, ... , each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence? *Source: AMC*

16. $\star\star$ Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. Find a_{12}. *Source: AMC*

17. $\star\star$ If $a_1 = a_2 = 1$ and $a_{n+2} = \dfrac{a_{n+1} + 1}{a_n}$ for $n \geq 1$, compute a_t where $t = 1998^5$. *Source: ARML*

18. $\star\star$ Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$? *Source: AMC*

19. ⋆⋆ 5 people are playing in a golf tournament. If ties are allowed, how many possible final standings are there? One possible configuration is that the 2nd and 4th person tied for 1st, the 3rd person gets 3rd, and the 1st and 5th person tied for 4th.

20. ⋆⋆ Everyday at school, Jo climbs a flight of 6 stairs. Jo can take the stairs 1, 2, or 3 at a time. For example, Jo could climb 3, then 1, then 2. In how many ways can Jo climb the stairs? *Source: AMC 8*

21. ⋆⋆ There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical. *Source: AIME* Ans: 548

22. ⋆ ⋆ ⋆ A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

    - Any cube may be the bottom cube in the tower.
    - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

    Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000? *Source: AIME*

23. ⋆ ⋆ ⋆ Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy? *Source: AMC*

24. ⋆ ⋆ ⋆ A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible? *Source: AIME*

25. ⋆ ⋆ ⋆ A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. Find the probability that the player wins the game (by emptying the bag). *Source: AIME*