Iowa City Math Circle Handouts
Angles and Triangles

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July 25, 2020

1 Angles

We start off with a few basic definitions.

Definition. A *ray* is a line segment extended infinitely in one direction.

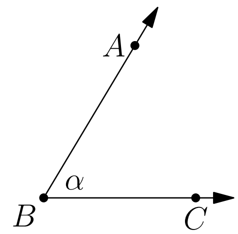
One may think of a ray as 'half of a line', as it only has one endpoint.

Definition. An *angle* as the union of two rays that share an endpoint. We all this common endpoint the *vertex* of the angle and refer to each of the two rays as a *side* of the angle. We can also use line segments as part of a side of an angle.

From our definitions, we see that we can define an angle by its vertex and two points, each on one of the sides. Conventionally, an angle $\angle XYZ$ is an angle with X on one side, Y being the vertex, and Z on the other side. Note that we symbolically denote an angle using the $\angle$ symbol.

Consider the following angle. We refer to this angle as $\angle ABC$, with vertex B and sides $\overrightarrow{BC}$ and $\overrightarrow{BA}$. For convenience, if there is a symbol in the interior of the angle near the vertex, then we refer to the angle by that name. For example, we can also call the following angle $\angle\alpha$. There are many ways we can measure an angle; the two most common angle measures are degrees and radians (which we will not discuss in this chapter). Here, we will assume that all angles are measured in degrees, which is denoted by a circle superscript. An angle whose two sides form a line has measure $180°$ and is called a *straight angle*. Additionally, an angle with half this measure (i.e. one with measure $90°$) is called a *right angle*.

It's most common for angles to measure between $0°$ and $180°$, but we can have angles measuring between $180°$ and $360°$. This is called a *reflex angle*, which is formed by



measuring the outside of an angle with standard measure less than 180°. For example, the reflex angle $\angle ABC$ in the above figure has measure of $360-\alpha$. Sidenote: to measure two angles, you can use a protractor, which you may think of as a handy "angular ruler."

Let's introduce a few more terms regarding angles.

Definition. Two angles with the same measure are called congruent angles.

Definition. Two angles are *complementary* if their measures sum to 90°. Additionally, two angles are *supplementary* if their sum is 180°.

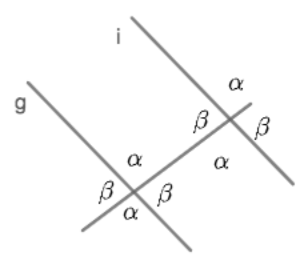
Definition. An *angle bisector* is a line through an angle that splits it into two congruent angles.

Definition. Any two intersecting lines that form an angle measuring 90° (i.e. a right angle) are called *perpendicular lines*. The symbol $\perp$ is used to denote perpendicularity, similar to the parallel symbol $\|$ being used to indicate parallel lines.

Definition. An angle with measure strictly between 0° and 90° is called *acute*, whereas an angle with measure strictly between 90° and 180° is called *obtuse*.

We'll get back to some of these definitions later in the chapter.

If a line intersects a pair of parallel lines (i.e. two lines that never intersect), then there are several pairs of congruent angles:



All angles marked with α are congruent, and all angles marked with β are congruent. We call the line intersecting both of the parallel lines a *transversal*. We see that when we draw two intersecting lines, we form two pairs of congruent angles. We refer to each pair as a pair of *vertical angles*, as the two angles share a common vertex but not any sides, making them appear vertically opposite. We can also write the equation $\alpha + \beta = 180$ (i.e. α and β are supplementary), as there exists several pairs of angles with measures α and β that form a pair of straight angles. We also get the following intuitive theorem, which gives us a criterion for determining whether two lines are indeed parallel.

Theorem 1.1. *Two lines are parallel if and only if any transversal (or segment) intersecting both lines is incident at the same angle.*

2 Triangles and Basic Properties

A triangle is a polygon with three sides and three angles. It is the only type of polygon where you can determine a unique triangle with only the side lengths.

Theorem 2.1. *The sum of the interior angles in a triangle is* $180°$.

Proof. In a triangle ABC, draw a line DE such that it is parallel to AB and passes through C. For this proof, we will have D on the A side of the triangle and E on the B side. Now, we can see that angle CAB equals angle DCA and angle CBA equals angle ECB. Because of line DE, we know that $\angle ECD + \angle CAB + \angle DCA = 180°$, so we can substitute such angles to get $\angle CBA + \angle CAB + \angle CBA = 180°$, which completes our proof. □

Checkpoint 2.1. Prove that the sum of the exterior angles(the supplementary angle of an interior angle) of a triangle is 360. Bonus: try proving this for any n-sided polygon!

The side lengths of a triangle can also correlate to the angles of a triangle.

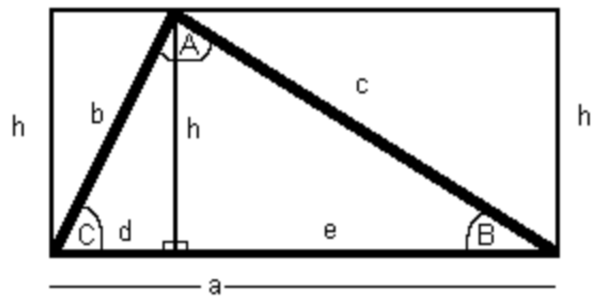
Theorem 2.2. *If $\angle A > \angle B > \angle C$, then $a > b > c$, where $a = BC, b = CA$, and $c = AB$.*

This can be proved with the Law of Sines, where $\frac{a}{\sin A} = \frac{b}{\sin B}$.

Definition. The altitude of a triangle is a line segment that extends from one vertex of the triangle and hits the opposite side at a $90°$ angle. Every triangle has three altitudes - one from each vertex.

Theorem 2.3. *The area of any triangle is $\frac{1}{2}bh$, where h is the length of any altitude of the triangle and b is the length of the side that this altitude hits (i.e. the side opposite to the vertex associated with the altitude). h stands for height and b stands for base.*

Proof. To prove this theorem, we enclose the triangle in a rectangle, where the longest side of the triangle coincides with the base of the rectangle. Now, we set the height of our rectangle such that the top edge of the rectangle will pass through the top vertex of our triangle. As you can see from the picture, the height of the rectangle turns out to be the same as the height of our triangle! (Can you see why? Note all the right angles in the picture).



Now, let us look at the two smaller rectangles in the picture, which arise from the altitude of the triangle splitting the bigger rectangle into two. We know that when we draw the diagonal of a rectangle, it splits the rectangle into two right triangles of equal area. In each of the smaller rectangles, half of each rectangle is included in the area of the triangle, as the diagonals of the smaller rectangles are the sides of the triangle. Therefore, the triangle takes up half of the area of the whole rectangle. The area of the whole rectangle is the product of its width and height, or ah (from the variables in the picture). Therefore, the area of the rectangle is $\frac{1}{2}ah$, where a is the base of the triangle. $\square$

This theorem can not only be used to find the area of a triangle, but can also be used to find the base/height of a triangle if we are given the area of the triangle (and sufficient additional information).

Checkpoint 2.2. A triangle has side lengths $4, 6$, and 7. Find the ratio between the longest altitude and the shortest altitude of the triangle.

We can classify triangles based on the types of angles they contain.

2.1 Right Triangles

Definition. A right triangle is a triangle that contains a right angle. The hypotenuse of a right triangle is the longest side of the triangle, namely, the side opposite the right angle. The two legs of a right triangle are the non-hypotenuse side; they are the two sides that are incident with the right angle.

Theorem 2.4. *For any right triangle, let c be its hypotenuse and a and b be its two legs. Then we have $a^2 + b^2 = c^2$, famously known as the Pythagorean Theorem.*

The Pythagorean Theorem tells us that given any two sides of a right triangle, we can always find the third.

Proof. The proof of the Pythagorean Theorem we choose to present (there are many different proofs of the theorem!) revolves around the picture below.
We can find the area of the big square in two ways. First, we can simply square the

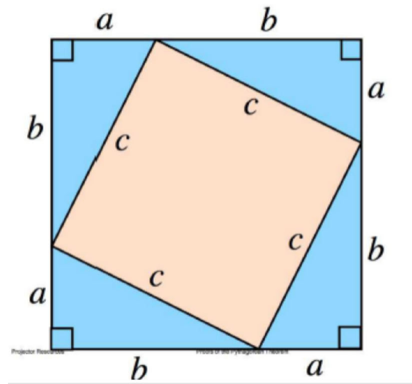
side length of the square, giving us an area of $(a+b)^2$. Second, we can sum up the areas of the four corner triangles and the middle square. The area of each triangle is $\frac{ab}{2}$ (as each triangle is just a right triangle with legs a and b), and the area of the middle square is c^2. This gives us a total area of $\frac{4}{2}ab + c^2 = 2ab + c^2$. Equating the two areas, we have

$$(a+b)^2 = 2ab + c^2$$

Expanding the LHS gives us $a^2 + 2ab + b^2$. Subtracting $2ab$ from both sides, we are left with

$$a^2 + b^2 = c^2$$

as desired. $\square$

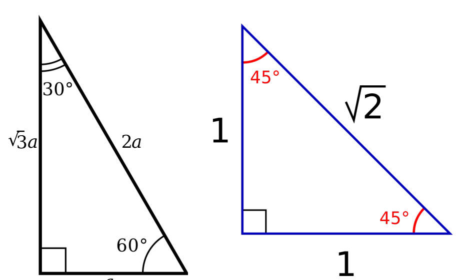


Checkpoint 2.3. Find the other leg of a right triangle with hypotenuse 41 and one leg 40.

Certain right triangles have special properties, and appear frequently in math competitions:

Definition. A $30-60-90$ triangle refers to any triangle with angles $30, 60,$and 90. The ratio between the hypotenuse and the shorter leg is 2 and the ratio between the longer leg and the shorter leg is $\sqrt{3}$, as shown in the figure below.

Definition. A $45-45-90$ triangle refers to any triangle with angles $45, 45,$and 90. Note that any such triangle is isosceles, as two of its angles are the same; in particular, the two legs of the right triangle are congruent. The ratio between the hypotenuse and any leg of the triangle is $\sqrt{2}$.



Checkpoint 2.4. Verify the ratios stated above for a $30-60-90$ triangle, by dropping down an altitude in an equilateral triangle and noting that this altitude splits the triangle into two $30-60-90$ triangles.

Theorem 2.5. *The area of a right triangle with legs a and b is $\frac{1}{2}ab$.*

Example 2.1. Consider a right triangle with legs a and b and hypotenuse c. Find the length of the altitude to the hypotenuse in terms of a and b.

Solution. Let us denote the length of the altitude to the hypotenuse by h. Now, lets find the area of the triangle in terms of a and b, which is $\frac{1}{2}ab$ since a and b are perpendicular. We can also write our area of the triangle with c and h, since they are perpendicular. With this, we have $\frac{1}{2}ch$, and we can equate this to our other area formula to get $\frac{1}{2}ch = \frac{1}{2}ab$, or $h = \frac{ab}{c}$. Finally, because we want the value of h only in terms of a and b, we can use the Pythagorean Theorem to write $c = \sqrt{a^2+b^2}$. This gives us $h = \frac{ab}{\sqrt{a^2+b^2}}$.

In this solution we have went over a very important technique - writing the area of a triangle in different ways (by choosing different base/height pairs) and equating these values. △

2.2 Equilateral and Isosceles Triangles

Lets start off with some definitions

Definition. A triangle is *isosceles* if it has two sides with the same length or two congruent angles.

In particular, the angles opposite to the equal sides will be congruent. These angles are called the *base angles.*

Theorem 2.6. *The altitude from the non-base angle divides isosceles triangle into two congruent triangles.*

This proof follows easily from congruence rules, which we will discuss later.

Corollary 2.6.1. *For an isosceles triangle, the altitude, median, and angle bisector from the non-base angle are equivalent.*

Definition. A triangle is equilateral if all three of its sides are the same length, or if all three of its angles are $60°$. If the triangle is equilateral, both of these conditions will be fulfilled.

With this information, lets now apply it to find useful formulas.

Solution. Find a general formula for the area of an equilateral triangle with side length s. △

Solution. We already know the base, so we only need to find the height of the triangle. In an equilateral triangle ABC, draw an altitude AD such that D is on BC. We now see that triangle ABD is a 30-60-90 right triangle. Here, we can now find AD since we know $BD = s/2$. Since $\frac{AD}{BD} = \sqrt{3}$ (30-60-90 triangle), $AD = \frac{\sqrt{3}s}{2}$. Now, we can use our $\frac{bh}{2}$ formula to get $s \cdot \frac{\sqrt{3}s}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{3}s^2}{4}}$. △

Checkpoint 2.5. Find the area of a regular hexagon with side length s (in terms of s).

3 Similarity and Congruence

Similarity and *congruence* are two very powerful tools for angle chasing. Let's first define what similarty and congruence means with respect to triangles.

Definition. Two triangles are *similar* when they have the same angles and their corresponding sides are in proportion.

Definition. Two triangles are congruent when they have the same angles and their corresponding sides have equal length.

One can say that two triangles are similar if and only if they are congruent up to a dilation. Similar triangles are just rotations, translations, reflections, and dilations of each other. All congruent triangles are similar, and the only additional constraint from similarity is that congruent triangles cannot be strict dilations of each other.

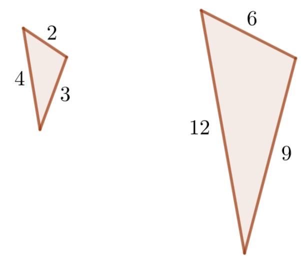
It turns out that we don't need to check that all sides are in proportion and that the angle measures are equal for similarity. There are special rules we can use to quickly tell whether triangles are similar. We can also use slight variations of these similarity rules to determine congruence. Next, we will introduce these rules and then show you how to apply them.

Proposition 3.1. *(SSS Similarity) Two triangles are similar if their corresponding sides are in proportion. This means that if*

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

then ΔABC is similar to ΔDEF.

The following two triangles are similar from SSS Similarity:

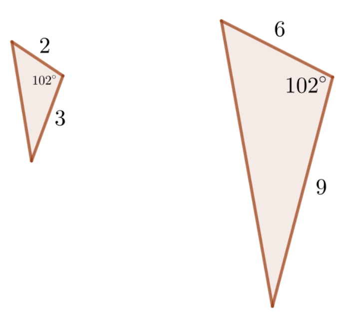


Proposition 3.2. *(SSS Congruence) If two triangles have the same side lengths, then they are congruent.*

In other words, two triangles having the same side lengths implies that the two also have the same angle measures; hence, they are congruent. The following two triangles are congruent by SSS Congruence:

Proposition 3.3. *(SAS Similarity) Two triangles are similar if two sets of sides are in proportion, and the angle between them are equal in the two triangles.*

The following two triangles are similar from SAS Similarity.

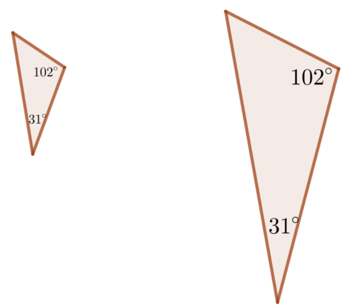


Proposition 3.4. *(SAS Congruence) If two triangles have two sides that are the same length and the angle between the two sides are equal, then the two triangles are congruent.*

The following triangles are congruent by SAS Congruence:

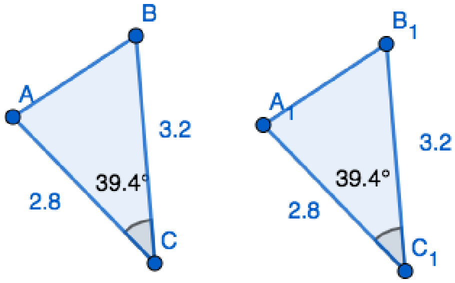
Proposition 3.5. *(AAA [or AA] Similarity) Two triangles are similar if they have the same angles. However, if we know two angles in a triangle, we know the third because the sum of the angles in a triangle is* 180°*. Hence, we only need that the two triangles have two angles in common (AA) in order to have similarity.*

Note that there is no AA congruence because the side lengths of the similar triangles can differ. The following two triangles are similar from AA Similarity:



Proposition 3.6. *(ASA Congruence) If two triangles have two angles that are congruent and if the sides in between the two angles have the same length, the two triangles are congruent.*

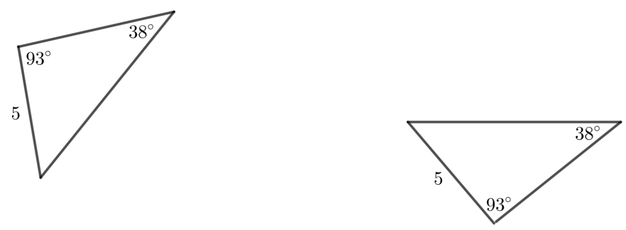
The following pair of triangles are congruent by ASA Congruence:



Note that ASA Similarity is the same as AA Similarity, because we have two congruent angles.

Proposition 3.7. *(SAA Congruence) If two triangles have two angles and a side length in common, then they are congruent. The common side must be in the same position relative to the two common angles.*

The following pair of triangles are congruent by SAA Congruence:

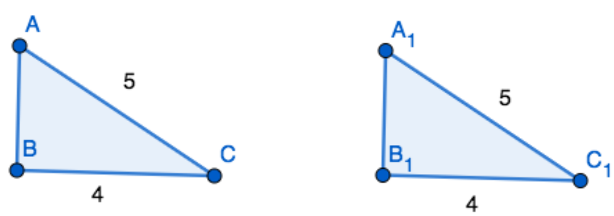


SAA Congruence is equivalent to ASA Congruence because you can find the measure of the other angle adjacent to the side length. We actually recommend ASA Congruence over SAA Congruence because SAA Congruence can get confusing. Again, SAA Similarity is the same as AA Similarity because we know that two of the angles are congruent.

Checkpoint 3.1. The following triangles are NOT congruent - why?

Proposition 3.8. *(HL Congruence) Two right triangles are congruent if they have a congruent hypotenuse and leg.*

For example, if $\angle ABC = \angle A_1B_1C_1 = 90°$, then ΔABC and $\Delta A_1B_1C_1$ are congruent by HL congruence because their hypotenuses are both 5, and one of their legs are 4.



If two right triangles' hypotenuse and leg are in proportion, then they will be similar. However, HL Similarity isn't as useful as HL Congruency.

Checkpoint 3.2. Why is HL Congruence true?

Example 3.1. In triangle ABC, M is the midpoint of AB and N is the midpoint of AC. Prove that the length of MN is $\frac{1}{2}BC$.

Solution. The key realization is that ΔAMN is similar to ΔABC in a $1:2$ ratio by SAS similarity. Hence, we get the desired result by using this similarity ratio to $\overline{MN}$. △

Example 3.2. Let $ABCD$ be a parallelogram. Show that ΔABC is congruent to ΔCDA using SSS, SAS, and SAA Congruence.

Solution. We know that $AB = CD$ and $BC = DA$, and since triangles ΔABC and ΔCDA share diagonal $\overline{AC}$, these triangles are similar by SSS. Similarly, using the fact that $AB \parallel CD$ and $BC \parallel DA$ (and establishing congruent angles by treating diagonal $\overline{AC}$ as a transversal) gives us that these two triangles are also similar by SAS and SAA. △

4 Angle Chasing

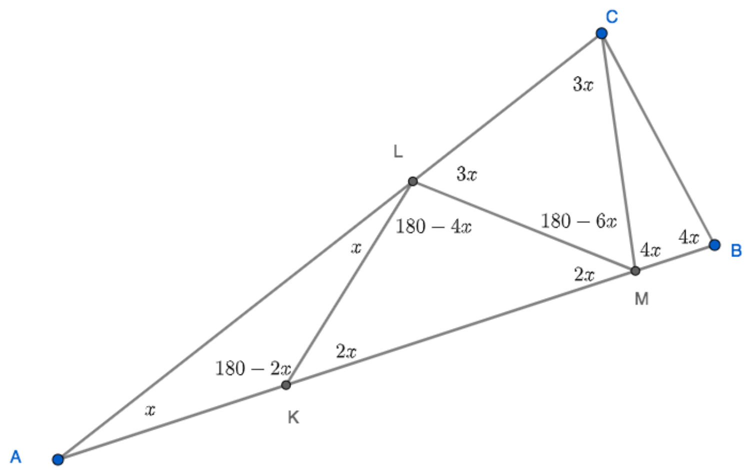
Angle chasing is the art of using geometric tools to find the measures of angles in a figure. The most helpful thing you can do to solve angle chasing problems (or any geometry problem) is to draw a large, neat diagram! We highly recommend getting a straightedge or a ruler to help keep your lines straight. Drawing tidy diagram will help you spot geometric figures or relationships easier, which could quickly unravel a problem or give you a hunch that guides how you will proceed. Some common examples of these include congruent segments and angles, similar and congruent triangles, right angles, isosceles and equilateral triangles, and parallel lines.

Another technique that aids with angle chasing is *geometric construction*. In this technique, we draw new auxiliary lines or extend some of the lines in the diagram in order to expose more of the geometry. Some common auxiliary lines to draw are parallel lines, angle bisectors, and altitudes.

The best way to master angle chasing is to solve many problems. Let's try an example which requires us to apply properties of isosceles triangles to angle chasing.

Example 4.1. Let ABC be a triangle with $AB = AC$ and let K and M be points on the side AB and L a point on the side AC such that $BC = CM = ML = LK = KA$. Find $\angle A$. *Source: 106 Geometry*

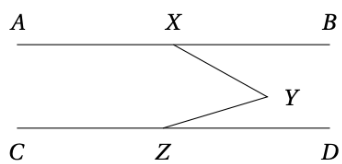
Solution. This problem requires drawing a good diagram and doing lots of angle chasing. If you let $\angle A = x$, you should obtain a diagram similar to the figure below through some basic angle chasing.



Using the fact that many of the triangles are isosceles, we eventually find that $\angle B = 4x$. Since triangle ABC is isosceles, $\angle B$ also equals $\frac{180-x}{2}$. This gives us the equation $\frac{180-x}{2} = 4x \implies x = \boxed{20^\circ}$. △

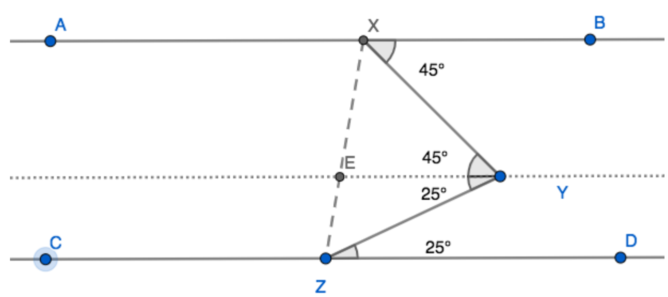
Let's take a look at another example that requires us to draw auxiliary lines.

Example 4.2. In the figure below, $\overline{AB}$ is parallel to $\overline{CD}$, $\angle BXY = 45^\circ$, $\angle DZY = 25^\circ$, and $XY = YZ$. What is the degree measure of $\angle YXZ$? *Source: Purple Comet*



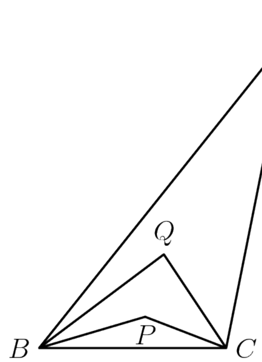
Solution.

We draw the line parallel to $\overline{AB}$ and $\overline{CD}$ through Y, and then construct $\overline{XZ}$. Let point E be the intersection of these two lines.



Now we know that $\angle ZYE = 25^\circ$ and $\angle XYE = 45^\circ$ through the parallel lines. Since ΔXYZ is isosceles, $\angle YXZ = \frac{180-(25+45)}{2} = \boxed{55^\circ}$. △

Checkpoint 4.1. The trisectors of angles B and C of scalene triangle ABC meet at points P and Q as shown. Angle A measures 39 degrees and angle QBP measures 14 degrees. What is the measure of angle BPC? *Source: MATHCOUNTS*

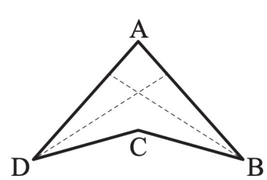


5 Problems

1. ⋆ Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD? *Source: AMC 12*

2. ⋆ Find x in the figure below:

3. ⋆ Concave quadrilateral $ABCD$ is symmetric about the line AC. The measures of angles DAB and ABC are 84 degrees and 32 degrees, respectively. The dashed line segments bisect angles ABC and ADC. What is the degree measure of the acute angle at which the two dashed line segments intersect? *Source: MATHCOUNTS*



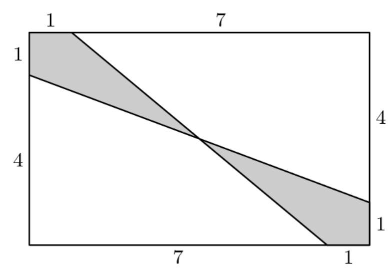
4. ⋆ The dimensions of a triangle are tripled to form a new triangle. If the area of the new triangle is 54 square feet, how many square feet were in the area of the original triangle? *Source: Alcumus*

5. ⋆ Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. What is the ratio of the area of the other small right triangle to the area of the square? Express your answer as a common fraction in terms of m. *Source: Alcumus*

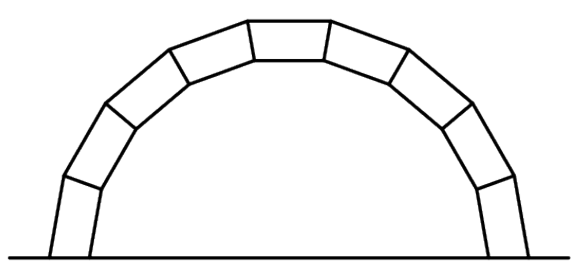
6. ⋆ Right triangle ABC has one leg of length 6 cm, one leg of length 8 cm and a right angle at A. A square has one side on the hypotenuse of triangle ABC and a vertex on each of the two legs of triangle ABC. What is the length of one side of the square, in cm? Express your answer as a common fraction. *Source: Alcumus*

7. ⋆ Let ΔABC be a triangle with $\angle A = 70°$. In addition, let D be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Given that $\angle ACD = 95°$ and $\angle ABD = 85°$, find the measure of $\angle ACB$.

8. ⋆ Find the area of the shaded region in the figure below. *Source: AMC*



9. ⋆⋆ The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



10. ⋆⋆ Let $ABDC$ be a quadrilateral with $AB = BC = CD$, $\angle B = 100°$ and $\angle C = 130°$. Find $\angle A - \angle D$.

11. $\star\star$ Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?*Source: AMC*

12. $\star\star$ In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC : DE = 4 : 3$. What is the ratio $AD : DB$? *Source: AMC*

13. $\star\star$ A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$? *Source: AMC*

14. $\star\star$ Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$? *Source: AMC*

15. $\star\star$ In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$? *Source: AMC 8*

16. $\star\star$ In $\triangle ABC$, a point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$? *Source: AMC 8*

17. $\star\star$ Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$? *Source: AMC 8*

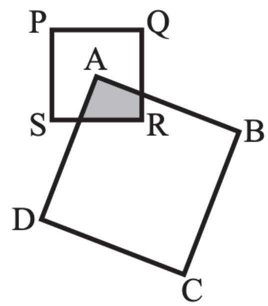
18. $\star\star$ In equiangular octagon $CAROLINE$, $CA = RO = LI = NE = \sqrt{2}$ and $AR = OL = IN = EC = 1$. The self-intersecting octagon $CORNELIA$ encloses six non-overlapping triangular regions. Find the area enclosed by $CORNELIA$, that is, the total area of the six triangular regions. *Source: AIME*

19. $\star\star$ Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2. *Source: AIME*

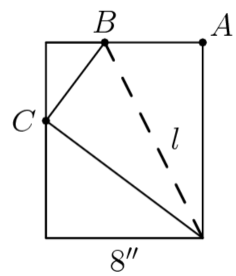
20. $\star\star$ Point P is on hypotenuse $\overline{EN}$ of right triangle BEN such that $\overline{BP}$ bisects $\angle EBN$. Perpendiculars $\overline{PR}$ and $\overline{PS}$ are drawn to sides $\overline{BE}$ and $\overline{BN}$, respectively. If $EN = 221$ and $PR = 60$, compute $\frac{1}{BE} + \frac{1}{BN}$. *Source: ARML*

21. $\star\star$ Two squares are placed, as shown, so that a vertex of the larger square, $ABCD$, is at the center of the smaller square, $PQRS$. If the squares have areas in the

ratio 3 : 1, what fraction of the area of square $PQRS$ is shaded? Express your answer as a common fraction. *Source: MATHCOUNTS*

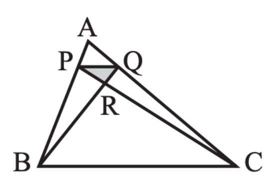


22. ⋆⋆ Corner A of a rectangular piece of paper of width 8 inches is folded over so that it coincides with point C on the opposite side. If $BC = 5$ inches, find the length in inches of fold l. *Source: AoPS*

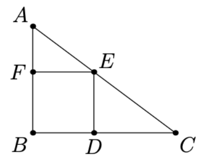


23. ⋆⋆ In triangle ABC, shown here, P and Q lie on sides AB and AC, respectively, so that $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{1}{5}$. Segments PC and QB intersect at R. What is the ratio of the area of triangle PQR to the area of triangle ABC? *Source: MATHCOUNTS*

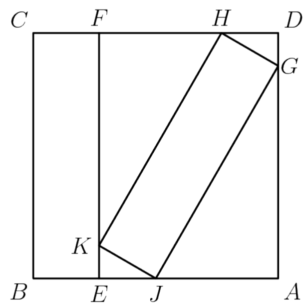
24. ⋆⋆ Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$? *Source: AMC 10*



25. ⋆⋆ The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$? *Source: AMC 10*



26. ⋆⋆ In the figure, $ABCD$ is a square of side length 1. The rectangles $JKHG$ and $EBCF$ are congruent. What is BE? *Source: AMC 12*



27. ⋆⋆ Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and $\overline{AC}$ intersects $\overline{XY}$ at point Z. Prove that

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

28. ⋆⋆ In the figure below, $BDEF$ is a square inscribed in ΔABC. If $\frac{AB}{BC} = \frac{4}{5}$, what is the area of $BDEF$ divided by the area of ΔABC? *Source: Math Prize For Girls*

29. ⋆ ⋆ ⋆ The degree measures of the angles of a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle. *Source: AIME*

30. ⋆⋆⋆ In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. Find the area of triangle ABC. (Hint: Draw the altitude of ΔABC from C) *Source: AIME*

31. ⋆ ⋆ ⋆ Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90°, AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?*Source: AMC*

32. ⋆⋆⋆ In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2? *Source: AMC*

33. $\star\star\star$ Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$, and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$? *Source: AMC*