$$i^4 = 1, \quad i^8 = 1 \ldots$$

①

$$12, 16, \ldots 96$$

$$3, 4, \ldots 24$$

$$24 - 3 + 1 = \boxed{22}$$

②

$$1-2i \qquad 9+ci$$

$$17^2 = \underbrace{(1-9)^2}_{=8} + (-2-c)^2$$

$$8 - 15 - 17$$

$$(-2-c)^2 = 17^2 \qquad \boxed{4}$$

$-2-c = 17 \Rightarrow c = -19 \qquad -2-c = -17 \Rightarrow c = 15$

③

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1-\frac{1}{1+i}}}}$$

$$1-\frac{1-i}{1+1}=\frac{1+i}{2}$$

$$1+\frac{2}{1+i}=1+\frac{2(1-i)}{2}=2-i$$

$$1+\frac{1}{2-i}=1+\frac{2+i}{4+1}=\frac{7+i}{5}$$

$$\frac{5}{7+i}\cdot\frac{7-i}{7-i}=\frac{35-5i}{49+1}=\frac{35}{50}-\frac{5}{50}i \qquad \to \boxed{\dfrac{3}{5}}$$

④ $$|z-2| = |z+1|$$

$$z = a + bi$$

$$|a - 2 + \underline{bi}| = |a + 1 + \underline{bi}|$$

$$(a-2)^2 = (a+1)^2$$

$$(a-2)^2 - (a+1)^2 = 0$$

$$(a-2+a+1)(a-2-a-1) = 0$$

$$2a = 1 \implies a = \frac{1}{2} \quad \boxed{\text{All } z \text{ with } Re(z) = \frac{1}{2}}$$

⑤

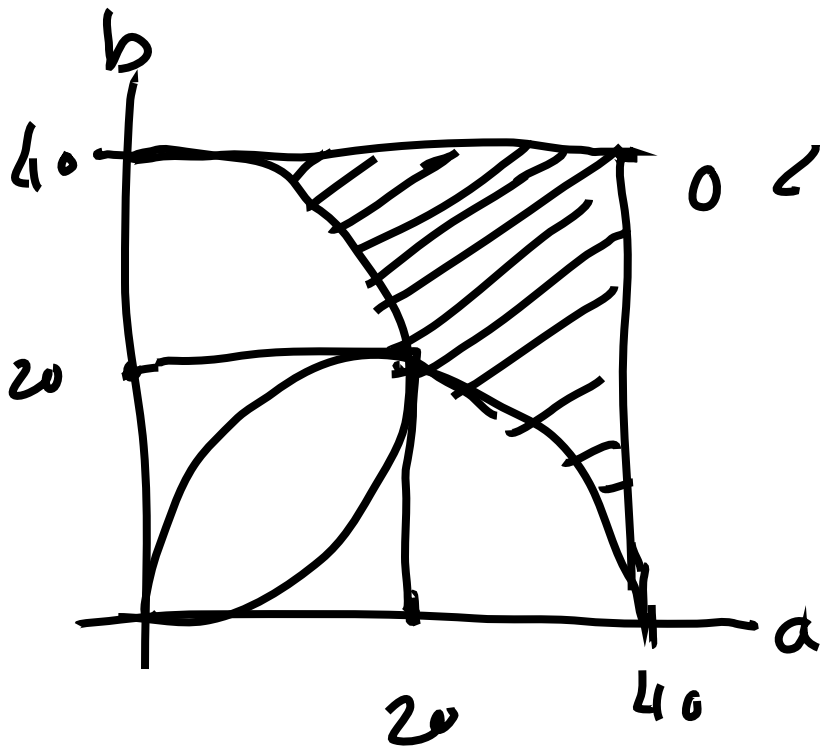$$a + a^{2020} = 0$$

$$a(1 + a^{2019}) = 0$$

$$a = 0 \qquad a^{2019} = -1$$

$$2019 + 1 = \boxed{2020}$$

⑥

$$z = a + bi$$

$$\frac{z}{40} = \frac{a}{40} + \frac{b}{40}i \; ; \; \frac{40}{\bar{z}} = \frac{40}{a - bi} = \frac{40a + 40bi}{a^2 + b^2}$$

$$0 < \frac{40a}{a^2 + b^2} < 1 \Rightarrow 40a < a^2 + b^2$$

$$0 < a^2 - 40a + b^2$$

$$20^2 = 400 < (a - 20)^2 + b^2$$

$$40^2 - 20^2 - \frac{1}{2} 20^2 \pi = \boxed{572}$$

⑦

$$(x^{15} - 1)(x^{15} + 1) = 0$$

$$x^{30} - 1 = 0$$

$$x^{30} = 1$$

$$\boxed{30}$$

⑧

$$C = (a+bi)^3 - 107i$$

$$C = a^3 + 3a^2bi - 3ab^2 - b^3i - 107i$$

$$C + 107i = a^3 - 3ab^2 + 3a^2bi - b^3i$$

$$C = a(a^2 3b^2) \qquad 107 = b(3a^2 - b^2)$$

$$C = 6(36 - 3) \qquad b = 1$$

$$= 6 \cdot 33 = \boxed{198} \qquad a = 6$$

⑨ $a = 20 + 40i$ $b = -14 + ci$

$$\frac{a}{b} = \frac{20 + 40i}{-14 + ci} = \frac{(20 + 40i)(-14 - ci)}{196 + c^2}$$

$$= -280 - 560i - 20ci + 40c$$

$$-280 + 40c = 0 \qquad \Rightarrow c = \boxed{7}$$

⑩

$$z^{24} = 1$$

$$z = cis\left(\frac{2\pi k}{24}\right)$$

$$z^6 = cis\left(\frac{6 \cdot 2\pi k}{24}\right)$$

$$= \cos\left(\frac{\pi k}{2}\right) + i\sin\left(\frac{\pi k}{2}\right)$$

$$k = 0, 2, 4, 6 \cdots$$

$$\boxed{12}$$

(11)

$$z^{12} = 64 \qquad 64^{1/12} = \sqrt{2}$$

$$z = \sqrt{2}\left(\cos\left(\frac{2\pi k}{12}\right) + i\sin\left(\frac{2\pi k}{12}\right)\right)$$

$$k = 0, 1, 2, \cancel{3, 4, 5, 6, 7, 8, 9}, 10, 11$$

$$\sqrt{2} + 2\left(\sqrt{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \sqrt{2}\right)$$

$$= \boxed{2\sqrt{2} + \sqrt{6}}$$

⑫

z z^{2019} θ

$$\frac{\pi}{\text{III}} \Bigg| \frac{\text{I}}{\text{IV}}$$

$\alpha = \frac{\pi}{4038}$ $\frac{\pi}{2} < \theta < \pi$

$$\boxed{2019\alpha < \theta < 4038\alpha}$$

$$\frac{\pi}{2} + 2\pi k < 2019\theta < \pi + 2\pi k$$

$$\frac{\pi}{2}(1 + 4k) < 2019\theta < \frac{\pi}{2}(2 + 4k)$$

$$\alpha(1 + 4k) < \theta < \alpha(2 + 4k)$$

$1009 - 505 + 1 = 505$

$(2021\alpha, 2022\alpha)$ 2020 505

$(2025\alpha, 2026\alpha)$ 2024 506

$\vdots$ $\vdots$ $\vdots$

$(4037\alpha, 4038\alpha)$ 4036 1009

$$\frac{505\alpha}{2\pi} = \frac{505}{2.4438} = \boxed{\frac{505}{8076}}$$

(13)

$$z^{13} = w \quad , \quad w^{11} = z$$

$$\left(z^{13}\right)^{11} = z$$

$$\Rightarrow \quad z^{143} = z \quad \Rightarrow \quad z\left(z^{142} - 1\right) = 0$$

$$z = \text{cis}\left(\frac{2\pi k}{142}\right) = \text{cis}\left(\frac{2\pi k}{71}\right)$$

$$\boxed{71}$$