Week 2:
Complex Numbers

$i = \sqrt{-1}$. In other words, $i^2 = -1$.

Write $\sqrt{-9}$ in terms of i.

Calculate the roots of $x^2 + 4x + 16$.

Identify the real and imaginary parts of the complex number $3 - 4i$.

Evaluate $1 + i + i^2 + i^3 + \cdots + i^{2019} + i^{2020}$.

Checkpoint 1.5. Evaluate r and θ (namely the magnitude and the argument) of the complex number $5 + 5i$.

Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $w = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. Calculate $z - w$, $\frac{z}{w}$, and $z \cdot w$.

The conjugate of a complex number $z = a + bi$: $\overline{z} = a - bi$

Also note that $(a - bi)(a + bi) = a^2 + b^2$.

1. $Re(z) = \frac{z+\overline{z}}{2}$

2. $Im(z) = \frac{z-\overline{z}}{2i}$

3. $z \cdot \overline{z} = |z|^2$

(a) *For* $|z| = 1$ *(i.e. complex numbers on the unit circle), we have* $\overline{z} = \frac{1}{z}$

4. A complex number z is real if and only if $z = \overline{z}$

5. $\overline{w} \cdot \overline{z} = \overline{wz}$ *and* $\overline{w}/\overline{z} = \overline{w/z}$ *for* $z \neq 0$.

Write $\frac{1}{3-5i}$ in the form $a+bi$, where a and b are real numbers.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \textit{ for all integers } n.$$

Calculate the magnitude and argument of $\left(\sqrt{2}\text{cis}(\frac{\pi}{8})\right)^{20}$

Theorem 1.3. *The n-th roots of $z = r(\cos\theta + i\sin\theta)$ are given by*

$$r_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

Find the fourth roots of $z = -1 + \sqrt{3}i$.

Corollary 1.3.1. *The n-th roots of unity are given by*

$$r_k = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1 \ldots n-1$.

Example 1.1. How many nonzero complex numbers z have the property that 0, z, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle? *Source: AMC 12*