Iowa City Math Circle Handouts
Complex Numbers Problems

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1 Definitions

A complex number is a number of the form $z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Notice that every complex number z can be decomposed into a real part, denoted by $\text{Re}(z)$, and an imaginary part, denoted by $\text{Im}(z)$. From our above definition (which we call the rectangular form of expressing a complex number), we have $\text{Re}(z) = a$ and $\text{Im}(z) = b$. This gives us the sense that a complex number z can be represented as the point (a, b) in the complex plane, with the horizontal axis representing the real part and the vertical axis representing the imaginary part.

Checkpoint 1.1. Write $\sqrt{-9}$ in terms of i.

Checkpoint 1.2. Calculate the roots of $x^2 + 4x + 16$.

Checkpoint 1.3. Identify the real and imaginary parts of the complex number $3 - 4i$.

Checkpoint 1.4. Evaluate $1 + i + i^2 + i^3 + \cdots + i^{2019} + i^{2020}$.

Perhaps a more useful representation of a complex number is $z = r(\cos\theta + i\sin\theta)$, where r is a non-negative real number and $\theta \in [0, 2\pi]$ radians. Note that $\cos\theta + i\sin\theta$ is often shorthanded by $\text{cis}\theta$. In this sense, r is thought of as the magnitude of the complex number, where as θ describes the angle. Since we can now plot this complex number easily on the polar plane, we call this form the polar form of a complex number. Comparing the polar form to the rectangular form, we see that $a = r\cos\theta$, $b = r\sin\theta$, and $r = \sqrt{a^2 + b^2}$. Both forms have certain advantages. For example, we prefer the rectangular form when we are adding/subtracting complex numbers or visualizing them in the coordinate plan. In contrast, the polar form is preferable when we are dealing with the multiplication/division of complex numbers. We'll explore the preceding remarks in detail in the following section.

Checkpoint 1.5. Evaluate r and θ (namely the magnitude and the argument) of the complex number $5 + 5i$.

2 Complex Number Arithmetic

First, let's see how we add two complex numbers. In rectangular form, we have

$$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i.$$

From this, we get that $\mathrm{Re}(z_1 \pm z_2) = \mathrm{Re}(z_1) \pm \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1 \pm z_2) = \mathrm{Im}(z_1) \pm \mathrm{Im}(z_2)$. As one may suspect, addition doesn't work out as nicely in polar form because there's no simple form for the addition of trigonometric functions. Now, we shift our focus to multiplication. In rectangular form, we have

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

From this, we get that

$$\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\mathrm{Im}(z_2)$$

and

$$\mathrm{Im}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Im}(z_2) + \mathrm{Re}(z_2)\mathrm{Im}(z_1).$$

We see multiplication of complex numbers in rectangular isn't all that nice. Now, let's experiment with multiplication in polar form. Using the cosine and sine addition formulae, we have

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i\right] \\ &= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i\right]. \end{aligned}$$

This looks wonderful! Complex number multiplication in a nutshell! From the above calculations, we see that $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. In the previous statements, we use the notation $|z|$ to represent the magnitude r and $\arg(z)$ to represent the angle θ. Finally, let's turn ourselves to the division of complex numbers. Let's first try it with complex numbers in rectangular form:

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} = \left(\frac{a_1 + b_1 i}{a_2 + b_2 i}\right)\left(\frac{a_2 - b_2 i}{a_2 - b_2 i}\right) = \frac{(a_1 a_2 + b_1 b_2) + (b_1 a_2 - a_1 b_2)i}{a_2^2 + b_2^2}$$

From this, we conclude

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{\mathrm{Re}(z_1)\mathrm{Re}(z_2) + \mathrm{Im}(z_1)\mathrm{Im}(z_2)}{|z_2|^2}$$

and

$$\mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{\mathrm{Im}(z_1)\mathrm{Re}(z_2) - \mathrm{Re}(z_1)\mathrm{Im}(z_2)}{|z_2|^2}.$$

This is clearly messy; let's see if this comes out cleaner if we use the polar form of complex numbers.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)\left(\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right) \\
&= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2) \\
&= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)i\right] \\
&= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]
\end{aligned}$$

This is quite concise! Furthermore, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$. In conclusion, we have seen how to do arithmetic with complex numbers using the two different forms discussed in the first section. Since some operations are easier in one form than another, it is key that you can not only recognize and convert between these two forms, but also that you have great facility doing algebra with them. We will expound on this in the following section.

Checkpoint 2.1. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $w = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. Calculate $z - w$, $\frac{z}{w}$, and $z \cdot w$.

3 Complex Conjugates and Basic Complex Number Results

The conjugate of a complex number $z = a + bi$ is defined as $\overline{z} := a - bi$. For example, the conjugate of $3 + 4i$ is $3 - 4i$. From this, we see that the complex conjugate of any real number is just itself, whereas the complex conjugate of any purely imaginary number is just it's negative. There are many useful properties of conjugates, as we will see in the following theorem.

Theorem 3.1. *We have the following properties involving complex conjugates:*

1. $Re(z) = \frac{z+\overline{z}}{2}$
2. $Im(z) = \frac{z-\overline{z}}{2i}$
3. $z \cdot \overline{z} = |z|^2$
   (a) *For* $|z| = 1$ *(i.e. complex numbers on the unit circle), we have* $\overline{z} = \frac{1}{z}$
4. *A complex number* z *is real if and only if* $z = \overline{z}$
5. $\overline{w} \cdot \overline{z} = \overline{wz}$ *and* $\overline{w}/\overline{z} = \overline{w/z}$ *for* $z \neq 0$.

Proof. Define $z = a + bi$ and $w = c + di$.

1. We have that $\frac{z+\overline{z}}{2} = \frac{(a+bi)+(a-bi)}{2} = \frac{2a}{2} = a = Re(z)$.

2. We have that $\frac{z-\overline{z}}{2i} = \frac{(a+bi)-(a-bi)}{2i} = \frac{bi}{2i} = b = Im(z)$.

3. We have

$$\begin{aligned} z \cdot \overline{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - (bi)^2 \\ &= a^2 - b^2 i^2 \\ &= a^2 - (-1)b^2 \\ &= a^2 + b^2 \\ &= \left(\sqrt{a^2+b^2}\right)^2 \\ &= |z|^2. \end{aligned}$$

4. We first prove that if z is real, then $z = \overline{z}$. Since z is real, we must have $b = 0$. Hence, $z = a + 0 \cdot i = a - 0 \cdot i = \overline{z}$.

   Now, we must prove the other direction: given $z = \overline{z}$, we must conclude that z is real. The equality $z = \overline{z}$ implies that $a + bi = a - bi$. Subtracting a from both sides, we are left with $bi = -bi$, or $b = -b$. The only real number that satisfies this is $b = 0$. Hence, $z = a + 0 \cdot i = a$, so z is real.

5. First, we will show that $\overline{w} \cdot \overline{z} = \overline{wz}$. We have

$$\begin{aligned} \overline{w} \cdot \overline{z} &= (c-di)(a-bi) \\ &= ac - bd + i(-ad - bc) \\ &= \overline{ac - bd - i(-ad - bc)} \\ &= \overline{ac - bd + i(ad + bc)} \\ &= \overline{(a+bi)(c+di)} \\ &= \overline{wz}. \end{aligned}$$

For the second part of the statement, we have

$$\begin{aligned}
&\overline{w}/\overline{z} \\
&= \frac{a-bi}{c-di} \\
&= \frac{(a-bi)(c+di)}{(c-di)(c+di)} \\
&= \frac{ac+bd+i(ad-bc)}{c^2+d^2} \\
&= \overline{\frac{ac+bd-i(ad-bc)}{c^2+d^2}} \\
&= \overline{\frac{(a+bi)(c-di)}{(c+di)(c-di)}} \\
&= \overline{\frac{(a+bi)}{(c+di)}} \\
&= \overline{\frac{w}{z}},
\end{aligned}$$

where $z \neq 0$ (as we cannot have division by 0).

□

These properties show that using/introducing complex conjugates can really be useful to simplify complex number expressions. Furthermore, we'll see later in the chapter the meaning of the conjugate geometrically and how that can be useful.

Checkpoint 3.1. Write $\frac{1}{3-5i}$ in the form $a+bi$, where a and b are real numbers.

4 DeMoivre's Theorem and Euler's Formula

Let's see what happens when we raise a complex number z to a nonnegative power:

$$z^2 = [r(\cos\theta + i\sin\theta)]^2 = r^2\left[\left(\cos^2\theta - \sin^2\theta\right) + (2\sin\theta\cos\theta)i\right] = r^2\left[\cos(2\theta) + \sin(2\theta)i\right]$$

$$z^3 = [r(\cos\theta + i\sin\theta)]^3 = r^3\left[\cos^3\theta + \sin\theta\cos^2\theta i - \sin^2\theta\cos\theta - \sin^3\theta i\right] = r^3\left[\cos(3\theta) + \sin(3\theta)i\right]$$

Above, we used some standard trigonometry results, but you don't need to follow the algebra completely to see a pattern. This pattern is summarized in the following theorem, which is called Demoivre's Theorem.

Theorem 4.1. $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ *for all integers* n.

Proof. First we prove the theorem for all nonnegative integers n. We induct on n. Our base case is $n = 0$, which is trivially true. Now, let's assume that the inductive statement holds for $n = k$, and we aim to show that it also holds for $n = k + 1$. We perform the following algebraic manipulations

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= [\cos(k\theta) + i\sin(k\theta)](\cos\theta + i\sin\theta)\\ &= [\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta] + [\sin\theta\cos(k\theta) + \sin(k\theta)\cos\theta]i\\ &= \cos((k+1)\theta) + i\sin((k+1)\theta).\end{aligned}$$

Hence, we have showed DeMoivre's Theorem for all nonnegative integers n.
To show it for negative integers of the form $-n$, where n is positive, we first write

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \left[(\cos\theta + i\sin\theta)^n\right]^{-1}\\ &= \left[\cos(n\theta) + i\sin(n\theta)\right]^{-1}.\end{aligned}$$

Next, we rewrite the result $z\bar{z} = |z|^2$ as $z^{-1} = \frac{\bar{z}}{|z|^2}$. Now, we set $z = \cos(n\theta) + i\sin(n\theta)$. We see that $|z| = 1$. Furthermore, $\bar{z} = \cos(-n\theta) + i\sin(-n\theta)$. Hence,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \left[\cos(n\theta) + i\sin(n\theta)\right]^{-1}\\ &= \frac{\cos(-n\theta) + i\sin(-n\theta)}{1^2}\\ &= \cos(-n\theta) + i\sin(-n\theta),\end{aligned}$$

as desired. Therefore, we have proven DeMoivre's Theorem in its entirety. □

We can now extract the magnitude and argument of z^n from DeMoivre's Theorem to get the following corollary.

Corollary 4.1.1. *From DeMoivre's Theorem, we have that* $|z^n| = |z|^n$ *and* $arg(z^n) = n \cdot arg(z)$.

In our dealings with DeMoivre's Theorem, we saw that

$$(f(\theta))^n = f(n\theta),$$

where f is the polar form of the complex number as a function of θ. This property is also satisfied by another function, namely $f(x) = e^x$. Extending our above remarks to the exponential function, we see that any complex number can be written in the form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

This is known as Euler's Formula and gives rise to the famous equation

$$e^{i\pi} = -1,$$

which we get when plugging in $r = 1$ and $\theta = \pi$. This equation is so intriguing because it relates three of the most fundamental constants in math: e, i, and π. Moreover, Euler's Formula gives us another form of representing a complex number, which we call the exponential form. Now, we can trivially derive our rules for the multiplication/division of complex numbers and DeMoivre's Theorem.

Checkpoint 4.1. Calculate the magnitude and argument of $\left(\sqrt{2}\text{cis}(\frac{\pi}{8})\right)^{20}$

5 Complex Roots and the Roots of Unity

In prior sections, we have discussed many of the basic operations on complex numbers: addition, multiplication, and exponentiation to name a few. The following theorem gives us the n-th roots of a complex number, which we will see later are not only tremendously useful in solving algebraic problems but also provide us with a useful geometric tool.

Theorem 5.1. *The n-th roots of $z = r(\cos\theta + i\sin\theta)$ are given by*

$$r_k = r^{1/n}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$$

for $k = 0, 1 \dots n-1$.

Proof.

$$\begin{aligned}
(r_k)^n &= \left[r^{1/n}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)\right]^n \\
&= r\left(\cos\left(n\left(\frac{\theta+2\pi k}{n}\right)\right) + i\sin\left(n\left(\frac{\theta+2\pi k}{n}\right)\right)\right) \\
&= r\left(\cos\left(\theta+2k\pi\right) + i\sin\left(\theta+2k\pi\right)\right) \\
&= r(\cos\theta + i\sin\theta) \\
&= z
\end{aligned}$$

Furthermore, we know there these are the only n-th roots of z as the polynomial $x^n - z$ has at most n distinct roots. □

Checkpoint 5.1. Find the fourth roots of $z = -1 + \sqrt{3}i$.

Now, we turn our attention to the n-th roots of 1 (called the *roots of unity*). From the theorem, we get the following corollary, which essentially establishes the roots of unity.

Corollary 5.1.1. *The n-th roots of unity are given by*

$$r_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1 \dots n-1$.

One can observe that the n-th roots all lie on the unit circle and are equally spaced due to the sequence $\{\frac{2\pi k}{n}\}$ being an arithmetic sequence. Moreover, by connecting adjacent roots on the circle, we get a regular n-gon inscribed in the unit circle (note that this holds for the n-th roots of any complex number). Now, we examine the cube roots of unity, which are 1, $\frac{-1+\sqrt{3}i}{2}$, and $\frac{-1-\sqrt{3}i}{2}$. To derive results using the cube roots of unity, we denote $\omega = \frac{-1+\sqrt{3}i}{2}$. Using this, we have the following theorem consisting of two identities.

Theorem 5.2. *Let* $\omega = \frac{-1+\sqrt{3}i}{2}$. *Then*

- $\frac{-1-\sqrt{3}i}{2} = \omega^2$
- *The sum of the cube roots of unity is* 0.

Proof. By algebra, we have

$$\omega^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{(-1)^2 - 3 - 2\sqrt{3}i}{4} = \frac{-1-\sqrt{3}i}{2}$$

Thus, the square of a cube root of unity (besides 1) is equal to another cube root of unity.

Now, to prove the other identity, we see that a root of the quadratic x^2+x+1 is $\frac{-1\pm\sqrt{3}i}{2}$ from the quadratic formula. By substituting $x = \omega$ and seeing that this root is indeed a root of unity, we get the desired identity. □

These identities are quite useful not only when we are solving problems in complex numbers (as tested in the problems of this section), but also to determine the roots of a polynomial, which we will discuss in another chapter.

6 Complex Numbers in Geometry

In this section, we discuss the geometric properties of complex numbers when they are plotted on the complex plane. The *complex plane* has two axii: the x-axis is called the real axis, and the y-axis is called the imaginary axis. Any complex number $z = a + bi$ can be plotted on the complex plane as the point (a, b) in the normal Cartesian plane.

Now, there are some important quantities we can find by plotting complex numbers on the complex plane. One of them is the angle between two complex numbers. Suppose we have three complex numbers (a, b, and c) and we want to find the angle $\angle abc$. To do this, we first shift all of the complex numbers down by c in order to make c the origin. From this, we can see that the desired angle is given by $\arg(a-c) - \arg(b-c)$. This is simply equivalent to $\arg\left(\frac{a-c}{b-c}\right)$ from complex number division. In particular, the angle between two complex numbers with respect to the origin is just the argument of their ratio.

Another scenario we consider is when we rotated the complex number $z = r\text{cis}(\alpha)$ by θ radians counter-clockwise with respect to the complex number w in the complex plane. We proceed as in the previous scenario: we shift the complex numbers down by w so that we are rotating about the origin. Now, we use the fact that rotating a complex number by an angle around the origin just increases the argument of that complex number by the angle (this is easy to see from the definition of a complex number). Applying this, we have that the desired point formed by the rotation around w is simply $w + (z - w)\text{cis}\theta$. In particular, if we rotate z around the origin by θ degrees counter-clockwise, the resulting complex number would be just $r\text{cis}(\alpha + \theta)$ In other words, if we want to rotate a complex number by θ radians (counter-clockwise), we can just multiply it by the complex number $\text{cis}(\theta)$!

Now let's take a look at an application of the above scenarios. Imagine a right triangle, with vertices at z, w, and the origin such that the right angle of the triangle is at the origin. Then we have $w\text{cis}(\pm\frac{\pi}{2}) = rz$ for some real number r (in other words, a 90 degree rotation of w should result in a complex number that has the same argument as z). Simplifying, we have $\frac{w}{z} = \pm ri$, or $\frac{w}{z}$ is pure imaginary!

As we have discussed, the conjugate of a complex number of $z = a + bi$ is $\overline{z} = a - bi$. Let's plot these two complex numbers on the complex plane. We see that z is at (a, b) and $\overline{z}$ is at $(a, -b)$. Hence, the conjugate of a complex number is simply the reflection of the original complex number across the real axis. From this fact, we can also see that the product of any complex number and its conjugate will *always* be real. Specifically, if z has magnitude 1, we can take the reciprocal of the complex number to reflect it over the $y-$axis and produce it. This is a direct consequence of part 3 of Theorem 1.1.

Lets take a look at another problem that requires us to think complex numbers in a geometrical way.

Example 6.1. How many nonzero complex numbers z have the property that $0, z,$ and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle? *Source: AMC 12*

Solution. In order to solve this problem, we first need to understand the criteria needed for the three points to be in an equilateral triangle. For this, we can simply consider that the lines between 0 and z and 0 and z^3 need to be equal in length, and that the angle between each pair of the three lines is 60 degrees. Lets take a look at the first condition. The first condition implies that $|z| = |z^3|$. This implies that $|z| = |z^3| = 1$ from Corollary 1.2.1. Now, from simple angle chasing, the second condition requires that the angle between z and z^3 to be 60 degrees. This implies that $\arg\left(\frac{z^3}{z}\right) = \pm\frac{\pi}{3} + 2\pi k$, where k is any integer. Now, we can simplify the left-hand side as follows using Demoivre's Theorem:

$$\begin{aligned}\arg\left(\frac{z^3}{z}\right) &= \arg\left(z^2\right)\\ &= 2\arg(z)\end{aligned}$$

Hence,

$$\arg(z) = \frac{\pm\frac{\pi}{3} + 2\pi k}{2}.$$

Setting $k = 0$, we get the solutions $z = \text{cis}\left(\frac{\pi}{6}\right)$ and $z = \text{cis}\left(\frac{11\pi}{6}\right)$. Setting $k = 1$, we have the solutions $z = \text{cis}\left(\frac{7\pi}{6}\right)$ and $z = \text{cis}\left(\frac{5\pi}{6}\right)$. Note that if $k \geq 2$, the argument of z will be greater than 2π, so we avoid these cases. Therefore, we have 4 solutions in total. △

In summary, thinking of complex numbers in a geometrical way by using the complex plane is a quite powerful tool to solve complex number problems.

7 Problems

1. $\star$ How many two-digit positive integers n satisfy $i^n = 1$?

2. $\star$ What is the sum of all possible c such that $1 - 2i$ is 17 units away from $9 + ci$?

3. $\star$ If $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+i}}}}$ can be written in the form $a + bi$ (where a and b are real numbers), what is $a + b$?

4. $\star$ Find all complex numbers z such that $|z - 2| = |z + 1|$.

5. $\star$ Find the number of complex numbers a such that $a + a^{2020} = 0$.

6. $\star\star$ Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\bar{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A? *Source: AIME*

7. $\star\star$ How many unique solutions are there to $(x^{15} - 1)(x^{15} + 1)$?

8. $\star\star$ Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$. *Source: AIME*

9. $\star\star$ If $a = 20 + 40i$ and $b = -14 + c \cdot i$, and if $\frac{a}{b}$ is a purely imaginary number, what is c?

10. $\star\star$ There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number? *Source: AMC 12*

11. $\star\star$ What is the sum of the roots of $z^{12} = 64$ that have a positive real part? *Source: AMC 12*

12. $\star\star$ A complex number z is selected uniformly at random such that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane. *Source: ARML*

13. $\star\star$ The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin \frac{m\pi}{n}$, for relatively prime positive integers m and n with $m < n$. Find n. *Source: AIME*

14. $\star\star$ How many integers $n \geq 2$ are there such that whenever $z_1, z_2, ..., z_n$ are complex numbers such that

$$|z_1| = |z_2| = ... = |z_n| = 1 \text{ and } z_1 + z_2 + ... + z_n = 0,$$

then the numbers $z_1, z_2, ..., z_n$ are equally spaced on the unit circle in the complex plane?

15. $\star\star$ The solutions to the equation $(z+6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled A, B, and C. What is the least possible area of $\triangle ABC$? *Source: AMC 12*

16. $\star\star\star$ Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg\left(\frac{w-z}{z}\right)$. The maximum possible value of $\tan^2\theta$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. *Source: AIME*

17. $\star\star\star$ Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000. *Source: AIME*

18. $\star\star\star$ Let $z_1 = 18 + 83i$, $z_2 = 18 + 39i$, and $z_3 = 78 + 99i$, where $i = \sqrt{-1}$. Let z be the unique complex number with the properties that $\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z - z_2}{z - z_3}$ is a real number and the imaginary part of z is the greatest possible. Find the real part of z. *Source: AIME*

19. $\star\star\star$ How many complex numbers y such that $|y| = 1$ exist such that y, y^2, y^3, $y^4 ... y^{80}$ form an 80-sided regular polygon on the complex plane? The points do not need to be in order.

20. $\star\star\star$ Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m+n$. *Source: AIME*