Iowa City Math Circle

Saturday, May 30, 2020: Word Problems & Statistics

Example 7.1. Ellie has a pitcher with 8 ounces of 50% lemonade. How many ounces of 25% lemonade should she add to the pitcher to obtain 35% lemonade? (In general, an $n\%$ drink refers to one in which $n\%$ of its volume is concentrate, and the rest is water).

x = number of oz to add

$$0.5 \cdot 8 + 0.25 \cdot x = (8+x) \cdot 0.35$$

$$4 + 0.25x = 2.8 + 0.35x$$

$$1.2 = 0.1x$$

$$x = \boxed{12 \text{ oz}}$$

Example 7.2. A person buys a TI-84 calculator for a total of $86.24. Given that the calculator was bought at a 20% discount and that a 10 percent tax was applied, what was the original price of the calculator?

x = original price

$$1.1 \cdot 0.8x = 0.88x = 86.24$$

$$0.88 \overline{)86.24}$$

$$98.$$

$$x = \boxed{\$98}$$

Example 7.3. Marisa deposits 2000 dollars into her back account, and it grows at a rate of 3% every year (thanks to interest). How many years would it take for her account to reach \$2600 dollars?

$$1.03P = 0.03 \cdot P + P$$

t = years

$$2000 \cdot \underbrace{1.03 \cdot 1.03 \cdots 1.03}_{t \text{ times}}$$

$$2000 \cdot (1.03)^t = 2600$$

$$(1.03)^t = 2600/2000$$

$$t = \log_{1.03}\left(\frac{2600}{2000}\right)$$

Checkpoint 7.1. Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote. At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point? *Source: AMC*

X = total number of votes

$$0.65X - 0.35X = 90$$

$$0.30X = 90$$

$$\Rightarrow \quad X = \boxed{300}$$

Checkpoint 7.2. Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda? *Source: AMC*

m = movie ticket S = Soda

$$m = 0.20(A - S) \Rightarrow m = 0.2A - 0.2S$$

$$S = 0.05(A - m) \Rightarrow S = 0.05A - 0.05m$$

$$m = 0.2A - 0.2(0.05A - 0.05m)$$

$$m = 0.2A - 0.01A - 0.01m$$

$$0.99m = 0.19A$$

$$m = \frac{19}{99}A$$

$$S = 0.05\left(A - \frac{19}{99}A\right)$$

$$S = 0.05\left(\frac{80}{99}A\right)$$

Example 7.4. Ellie has a pitcher with 8 ounces of 50% lemonade. How many ounces of 25% lemonade should she add to the pitcher to obtain 35% lemonade? (In general, an $n\%$ drink refers to one in which $n\%$ of its volume is concentrate, and the rest is water).

Checkpoint 7.3. Ellie's lemonade stand has a pitcher with 4 cups of water. After adding in a saltwater solution to the pitcher, the pitcher now contains 9 cups of 40% saltwater. What is the concentration of the saltwater solution?

5 cups $\quad x$ = concentration

$9 \cdot 0.40 = 5 \cdot x$

$\Rightarrow \quad x = 0.72 \quad x = \boxed{72\%}$

Example 7.5. In a specific company, there are lumberjacks that need to cut trees. Assume that all lumberjacks cut down trees at the same rate, and there is no interference between lumberjacks. One day, 20 lumberjacks cut down 100 trees in 60 minutes. At this same rate, how long in minutes would it take for 50 lumberjacks to cut down 50 trees?

20 lumberjacks 100 trees → 60 min

$$60 \cdot \frac{1}{\frac{5}{2}} \cdot \frac{1}{2} = 60 \cdot \frac{2}{5} \cdot \frac{1}{2} = 12 \text{ minutes}$$

w = workers

t = time

s = size

6 farmers	3 farmers
6 chickens	12 chickens
6 minutes	time = ?

$$\frac{w_1 \cdot t_1}{s_1} = \frac{w_2 \cdot t_2}{s_2}$$

$$\frac{6 \cdot 6}{6} = \frac{3 \cdot t_2}{12} \Rightarrow t = \boxed{24}$$

Example 7.6. Reimu and Marisa are mowing the lawn, and assuming they work at constant rates. If Reimu were to mow the lawn alone, it would take her 20 minutes for her to complete the job. If both were working together mowing the lawn, then it would them 15 minutes to finish assuming both are working without interference. How long in minutes would it take for Marisa to mow the lawn if she works alone?

$\frac{1}{20}$

$\frac{1}{15}$

m $\frac{1}{m}$

$\frac{1}{20} + \frac{1}{m} = \frac{1}{15}$

$3 + \frac{60}{m} = 4$

$\frac{60}{m} = 1$

$\Rightarrow m = 60$

$$d = st \qquad t = \frac{d}{s} \qquad s = \frac{d}{t}$$

$$s_{avg} = \frac{d_{total}}{t_{total}} = \frac{d_1 + d_2 + d_3 + \cdots d_n}{t_1 + t_2 + t_3 + \cdots t_n}$$

$$= \frac{d_1 + d_2 + d_3 + \cdots}{\frac{d_1}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + \cdots}$$

Example 7.7. Kylie took a road trip in which she drove at 60 mph for the first 90 minutes, 40 mph for the next 60 miles, and covered the last 100 miles of her trip in 50 minutes. What was her average speed for the whole trip?

$d = 5t$ $\qquad s_{avg} = \frac{d}{t}$

$$s = \frac{60 \cdot 1.5 + 60 + 100}{1.5 + \frac{60}{40} + \frac{50}{60}}$$

$$\frac{90+60+100}{1.5+1.5+\frac{1}{2}} = \frac{250}{\frac{14}{6}} = \frac{1500}{14}$$

Checkpoint 7.4. Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?
Source: AMC

$0.5 \cdot 60 = 30$ miles

$0.5 \cdot 65 = 32.5$ miles

$96 - 62.5$ miles $= 33.5$ mi

$d = st$

$33.5 = s \cdot 0.5 \Rightarrow s = \boxed{67}$

Example 7.8. Kevin and Reece live in houses 90 miles apart (the distance between Cedar Falls and Iowa City). They both leave their houses and drive to the other person's house along a straight line. Kevin is driving at a constant speed of 75 mph, whereas Reece drives at a constant speed of 60 mph. How long after they start driving do they cross paths? Additionally, how far is the point at which they meet from each house?

$\frac{1}{60} \cdot 60 =$ 1 mile

$\frac{1}{60} \cdot 75 =$ 1.25 miles

2.25 miles

$\frac{90}{2.25} =$ 40 minutes

$\frac{2}{3} \cdot 60 =$ 40 miles

Example 7.9. A car drives from town A to B, a distance of 100 miles, at a speed of 60 miles per hour. At the same time the car leaves A, a fly leaves B in the direction of A, moving at a speed of 30 miles per hour. Upon meeting the car, the fly turns back around, and flies back to B. Once it reaches B, it turns around (in the direction of A), and repeats the process. Once the car reaches B, the fly stops moving. How much distance has the fly covered in total?

$$d = st$$

$$100 = 60t \qquad t = \frac{100}{60}$$

$$d = 30 \cdot \frac{100}{60}$$

$$= \boxed{150 \text{ miles}}$$

- The *mean*, or *average*, of a collection of values is the sum of the values divided by the number of values: the mean of the data collection $x_1, x_2, \cdots, x_n$ is $(x_1 + x_2 + \cdots + x_n)/n$. For example, the mean of the set $\{1, 3, 5, 6\}$ would be $(1+3+5+6)/4 = 3.75$.

- The *median* of a collection of numbers is the value that evenly separates the data into higher and lower values; in other words, the median is the number in the "middle" of the data when its sorted in ascending order. For example, the median of $2, 5, 8, 10, 11$ is 8. (When finding the median, make sure you place the numbers in *increasing* order first!) If there are an even number of values, the median is the average of the two middle values; for example, the median of $1, 4, 5, 7$ would be $(4 + 5)/2 = 4.5$.

- The *mode* is the value that occurs the most often in a data set. For example, 3 is the unique mode of the set $\{1, 2, 3, 3, 3, 4\}$. There can be multiple modes in a data set if the most frequent values occur with the same frequency. If a data set has only one mode, it is referred to as an *unique mode*.

- The *range* of a data set is the difference between the greatest and least values in the set. For example, the range of the set $\{1, 5, 9, 12\}$ is $12 - 1 = 11$.

Example 9.1. What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17$, and x is equal to the mean of those five numbers? *Source: AMC*

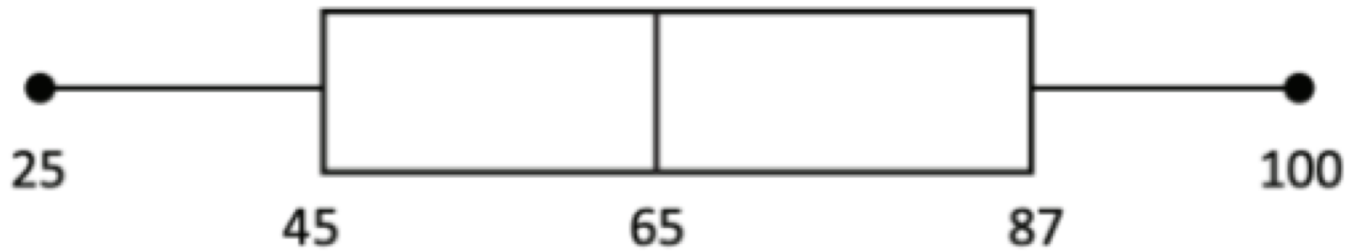
Checkpoint 9.1. The mean, median, and mode of the 7 data values $60, 100, x, 40, 50, 200, 90$ are all equal to x. What is the value of x? *Source: AMC* Ans: 90

Example 9.2. (Stem and Leaf Plots)

2	3 4 6 7
3	3 8 8 8 8
4	1 1 5

The stem-and-leaf plot shows the number of points scored by the winning team in each of the first 12 NFL games played this football season. What is the absolute difference between the mode and median of this data? *Source: MATHCOUNTS*

Example 9.3. (Box and Whisker Plots)



What is the positive difference between the range and the interquartile range of the data set represented by this box-and-whisker plot? *Source: MATHCOUNTS*

3. ⋆ The mean age of Amanda's 4 cousins is 8, and their median age is 5. What is the sum of the ages of Amanda's youngest and oldest cousins? *Source: AMC*
Ans: 22