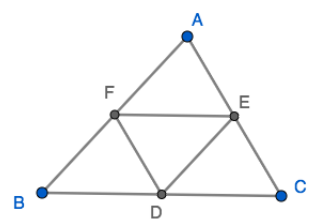
Chapter 30

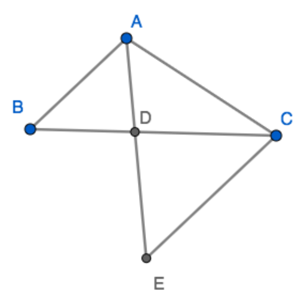
Solutions to Triangle Centers Problems

30.1 Checkpoints

1. This is fairly intuitive: all the triangles are congruent because they have the same angles and side lengths:



2. First, assume AD is the angle bisector of triangle ABC. We draw a line parallel to AB that passes through C and meets AD at E.



Because $AB||CE$, $\angle BAE = \angle CEA$. In addition, $\angle BAE = \angle CAE$ because AE bisects $\angle BAC$. Then triangle ACE is isosceles because $\angle CAE = \angle CEA$, so $AC = CE$. Triangle ABD is similar to triangle ECD, so by similarity,

$$\frac{AB}{BD} = \frac{EC}{CD} = \frac{AC}{CD},$$

as desired.

We'll now prove the theorem in the opposite direction (using the same diagram as above). Suppose that D is a point on BC so that $\frac{AB}{BD} = \frac{AC}{CD}$. Again, we draw a line parallel to AB that passes through C and meets AD at E. Triangles ABD and ECD are similar, so that gives us $\frac{AB}{BD} = \frac{EC}{CD}$. Then EC must equal AC, so triangle ACE is isosceles. That means $\angle CEA = \angle CAE$. Since $AB || CE$, $\angle BAD = \angle CEA = \angle CAE$, indicating that AD bisects $\angle BAC$.

Thus, we've shown that AD is the angle bisector through A if and only if $\frac{AB}{BD} = \frac{AC}{CD}$.

3. In ΔABC Let cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at point G inside the triangle. Since ΔAGF and ΔBGF have the same height, the ratio of their areas is $\frac{AF}{FB}$. Similarly, since ΔACF and ΔBCF have the same height, the ratio of their areas is $\frac{AF}{FB}$. Thus, by taking the difference of these ratios, we can conclude that the $\frac{[AGC]}{[BGC]} = \frac{AF}{FB}$. Using the same method applied to sides $\overline{BC}$ and $\overline{AC}$, we get that $\frac{[BAG]}{[AGC]} = \frac{BD}{DC}$ and $\frac{[BGC]}{[BGA]} = \frac{CE}{EA}$. By multiplying these three equations together, we get the desired result.

4. *Proof.* Let AD, BE, and CF be medians of a triangle ΔABC. The medians of a triangle divide each side into two congruent line segments. This means that $AF = FB$, $BD = DC$, $CE = EA$. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

. Thus by Ceva's Theorem, the medians of a triangle are concurrent at the centroid. □

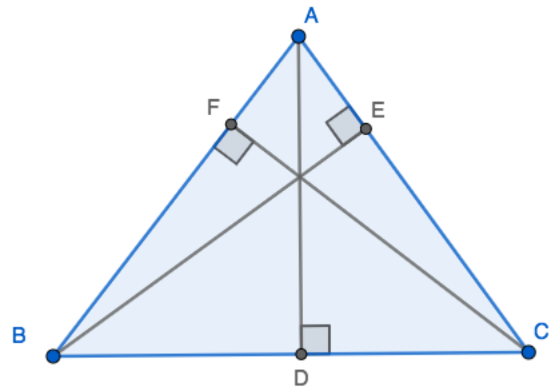
5. *Proof.* First we evaluate all the sines in

$$\frac{\sin(\angle DAB) \cdot \sin(\angle EBC) \cdot \sin(\angle FCA)}{\sin(\angle DAC) \cdot \sin(\angle EBA) \cdot \sin(\angle FCB)}$$

to get

$$\frac{\frac{BD}{AB} \cdot \frac{CE}{BC} \cdot \frac{AF}{AC}}{\frac{BC}{AC} \cdot \frac{AE}{AB} \cdot \frac{FB}{BC}} = \frac{BD \cdot CE \cdot AF}{DC \cdot AE \cdot FB}.$$

Note that the angular form of Ceva's Theorem isn't necessarily needed here — using the normal form will produce the same result.



Notice that right triangles ADC and BEC are similar due to AA Similarity. This means that $\frac{CE}{DC} = \frac{BE}{AD}$. Similarly (haha get it?), $\frac{AF}{FE} = \frac{FC}{BE}$ and $\frac{BD}{FB} = \frac{AD}{FC}$. We then take the expression from above and do some substitutions:

$$\begin{aligned}\frac{BD \cdot CE \cdot AF}{DC \cdot AE \cdot FB} &= \frac{BE}{AD} \cdot \frac{FC}{BE} \cdot \frac{AD}{FC} \\ &= 1\end{aligned}$$

Thus, all the altitudes must be concurrent. □

30.2 Exercises

1. The semiperimeter of the triangle will be $\frac{1}{2} \cdot 40 = 20$. Then using the area formula $A = rs$, we have $120 = r \cdot 20 \implies r = \boxed{6}$.

2. Let a, b, and c be the sides of the triangle, A be the area of the triangle, r be the inradius, and R be the circumradius. Using the fact that the triangle is a right triangle, we have that

$$\begin{aligned}R &= \frac{abc}{4A} \\ &= \frac{3 \cdot 4 \cdot 5}{4 \cdot \frac{3 \cdot 4}{2}} \\ &= \boxed{\frac{9}{2}}\end{aligned}$$

and

$$\begin{aligned}r &= \frac{A}{\frac{a+b+c}{2}} \\ &= \frac{\frac{3 \cdot 4}{2}}{\frac{3+4+5}{2}} \\ &= \boxed{1}\end{aligned}$$

3. Let $a = 13$, $b = 14$, and $c = 15$ be the sides of the triangle and s be the perimeter. We have that $s = \frac{13+14+15}{2} = 21$. The area of the triangle, by Heron's formula, is

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{21 \cdot 8 \cdot 7 \cdot 6} \\ &= \sqrt{2^4 \cdot 3^2 \cdot 7^2} \\ &= 2^2 \cdot 3 \cdot 7 \\ &= 84.\end{aligned}$$

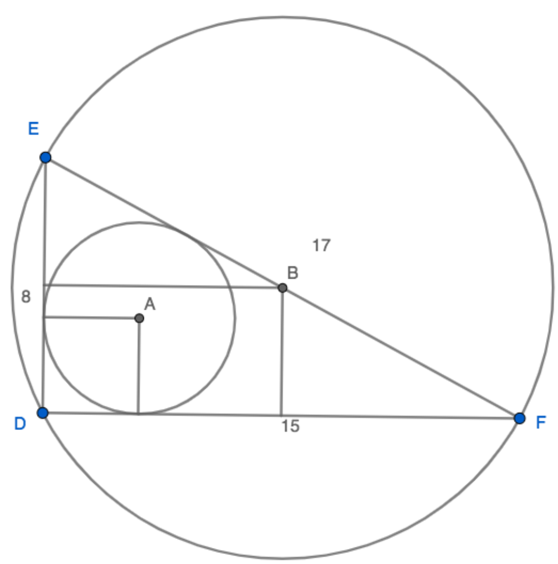
Thus, the circumradius of the triangle is

$$\frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \boxed{\frac{65}{8}}.$$

4. It's easy to see that the largest angle in this triangle is opposite the side with length 55 (the longest side). Notice that this is just a direct application of the Law of Sines. Let the bisector divide the side with length 55 into two segments with lengths x and y. We have that $x + y = 55$ and $\frac{33}{x} = \frac{44}{y}$ by the Angle Bisector Theorem. Thus, $y = \frac{4}{3}x$, so $x + \frac{4}{3}x = 55$. Solving for x (the smallest of these two segments) we have that $x = \boxed{\frac{165}{7}}$.

5. We reflect the triangle and the inscribed semicircle about line AC. This gives us an isosceles triangle with base length 10 and an inscribed circle. The area of this new triangle will be $\frac{1}{2} \cdot 12 \cdot 10 = 60$, and its semiperimeter will be $\frac{1}{2}(13+13+5) = \frac{31}{2}$. Then using the area formula, $A = rs \implies 60 = r \cdot \frac{31}{2} \implies r = \boxed{\frac{120}{31}}$.

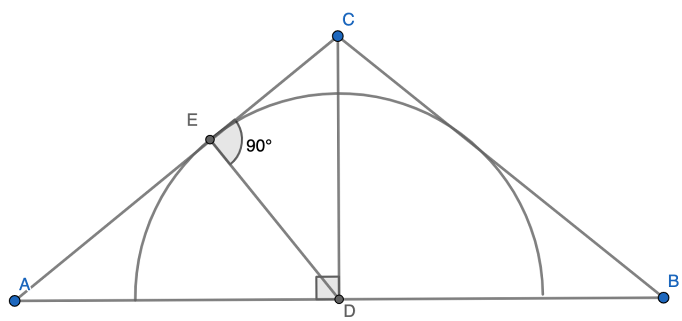
6. Since the side lengths of the triangle are 8,15, and 17, DEF is a right triangle.



That means the EF (in the diagram above) will be the diameter of the circumcircle, and the circumcenter will be the midpoint of that line. Now we'll find the length of the inradius. The area of ΔDEF is $\frac{1}{2}\cdot 8\cdot 15 = 60$, and the semiperimeter is $\frac{1}{2}(8+15+17) = 20$, so the inradius has length $\frac{60}{20} = 3$.

We'll now use the Distance Formula to calculate the length of AB. Since B is the midpoint of EF, its projection onto line DF will split DF into two congruent line segments of length 7.5. Thus the distance between A and B's x-coordinates will be $7.5 - 3 = 4.5$. Likewise, the distance between A and B's y-coordinates will be $4 - 3 = 1$. Thus, $AB = \sqrt{(\frac{9}{2})^2 + 1^2} = \boxed{\frac{\sqrt{85}}{2}}$.

7. Since the quadrilateral is equilateral (with side length 5), it must be a rhombus. We can slice this rhombus in half (along its diagonals) to get a semicircle inscribed in an isosceles triangle. We drop the altitude to the base and draw the radius to one of the points of tangencies, as shown below.

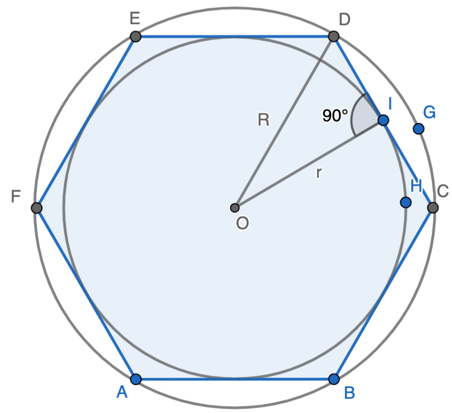


In the figure, ΔABC is isosceles with $AC = BC = 5$. CD is an altitude and E is the point of tangency of the semicircle to side $\overline{AC}$. Since $DE \perp AC$, we have that that

$$\begin{aligned}\frac{[ABC]}{2} &= [ADC] \\ &= \frac{AC \cdot DE}{2} \\ &= \frac{5}{2}DE\end{aligned}$$

Since $\overline{DE}$ is the radius of the semicircle (which is also the original inradius), the original inradius is maximized when the area of ΔABC is maximized. By the Law of Sines, the area of the triangle is given by $\frac{25 \sin \angle ACB}{2}$. Since $\sin 90^\circ = 1$, the area of the triangle is maximized when $\angle ACB = 90^\circ$. In other words, the original inradius is maximized when the figure is a square. In this case, the inradius is simply half the square's side length, or $\boxed{\frac{5}{2}}$.

8. Below is a diagram showing the circumcircle and incircle of a regular hexagon.



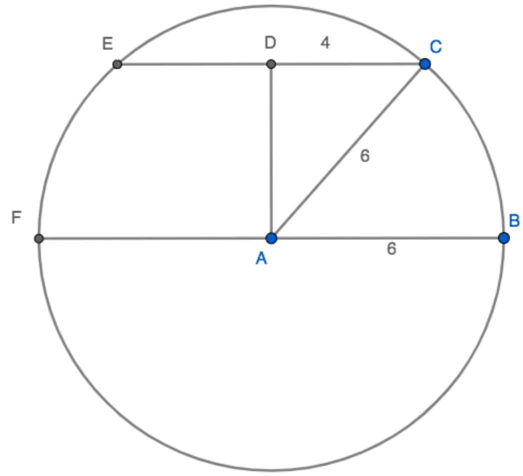
It is easy to see that if we draw $\overline{OC}$, ΔDOC is equilateral (use the fact that the interior angles of a regular hexagon measure 120°). Thus, $R = DC = 3$. Using the side ratios for a 30-6-90 right triangle, we have that $r = \frac{\sqrt{3}}{2}R$. Therefore,

$$R - r = R\left(1 - \frac{\sqrt{3}}{2}\right)$$
$$= \boxed{\frac{6 - 3\sqrt{3}}{2}}.$$

9. We can take the vertical cross section of the cone through its center. This gives us a circle inscribed in a triangle with a base of length 24 and a height of length 24. The area of the triangle will be $\frac{1}{2} \cdot 24 \cdot 24 = 288$, and the semiperimeter will be $\frac{1}{2}(24 + 2 \cdot 12\sqrt{5}) = 12 + 12\sqrt{5}$. Using $A = rs$, the radius of the inscribed circle will then be $\frac{288}{12+12\sqrt{5}} = 6\sqrt{5} - 6$. Since the radius of the circle is also the radius of the sphere, the radius of the sphere is $\boxed{6\sqrt{5} - 6}$.

10. First we take the cross section of the sphere at the plane where the triangle is. This is a triangle with an inscribed circle. Since the triangle is isosceles, we find its altitude from its apex is 9. This means it has an area of $\frac{1}{2} \cdot 9 \cdot 24 = 108$ and a semiperimeter of $\frac{1}{2}(15 + 15 + 24) = 27$. Using the area formula, the area of the inradius will be $r = 108/27 = 4$.

    We'll now take a cross section through the center of the sphere and perpendicular to the plane of the triangle.



In the figure above, EC represents the incircle of the triangle. We can draw radius FB so that $FB||EC$, then draw line AD perpendicular to EC. Using the Pythagorean Theorem, $AD = \sqrt{6^2 - 4^2} = 2\sqrt{5}$. AD is the distance from the center of the sphere to the plane containing the triangle, so the answer is $\boxed{2\sqrt{5}}$.