Chapter 29

Solutions to Coordinate Geometry Problems

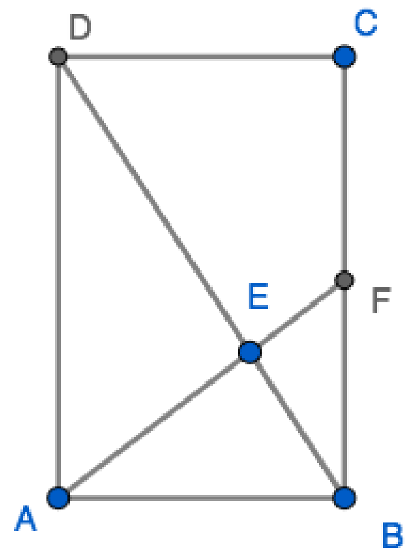
29.1 Warm-up Problems

1. Setting the equations equal to each other, we get $ax+b=cx+d$. Rearrange and solving for x, we get $x=\frac{d-b}{a-c}$. We plug this value in for x for one of the equations to find y: $y=a\cdot\frac{d-b}{a-c}+b=\frac{a(d-b)+b(a-c)}{a-c}$. Thus the intersection of the equations will be the point $\boxed{(\frac{d-b}{a-c},\frac{a(d-b)+b(a-c)}{a-c})}$.

2. We move the x term to the RHS and divide both sides by b to get the equation $y=-\frac{a}{b}\cdot x+\frac{c}{b}$. The slope is then the coefficient of x, which is $\boxed{-\frac{a}{b}}$.

29.2 Checkpoints

1. Triangles AED and FEB are similar by AA.



Since point E trisects the diagonal, $2BE = ED$. Due to similarity, $2BF = AD \implies 2BF = 2 \implies BF = 1$. From the Pythagorean Theorem, we then know $AF = \sqrt{2}$. Again due to similarity, E trisects AF, so $AE = \boxed{\dfrac{2\sqrt{2}}{3}}$.

2. *Proof.* We start with the fact that two lines are parallel iff they never intersect or they are the same line. So the original problem becomes to prove that $m_1 = m_2$ iff the two lines never intersect or they are the same line.

   First, we will show that if $m_1 = m_2$, then the lines never intersect or they are the same line. Let the first line be $y = m_1x + b_1$, and the second be $y = m_2x + b_2$. Subtracting the second equation from the first, we get $b_2 - b_1 = (m_1 - m_2)x$. Using the fact that $m_1 = m_2$, we have that $b_2 - b_1 = 0$. From this, we see that if $b_1 \neq b_2$, then $b_1 - b_2 \neq 0$, hence there are no solutions to the system of equations, equivalently, the lines do not intersect. If $b_1 = b_2$, then the two lines are the same as $m_1 = m_2$ and $b_1 = b_2$.

   Now, we will show that if the two lines never intersect or they are the same line, then the slopes of the two lines are equal. Let the first line be $y = ax + b$ and $y = cx + d$. Obviously, if they are the same line, $a = c$ so there slopes are equal. Now we will look at the case where these two lines never intersect. Subtracting the second from the first equation, we get $(a - c)x = d - b$. We see that if $a \neq c$, a solutions exists, where $x = \frac{d-b}{a-c}$ - hence, the lines intersect. Therefore, $a = c$, or the slopes of the lines are equal. □

3. *Proof.* Draw the auxiliary point (x_2, y_1) and draw the triangle with vertices (x_1, y_1), (x_2, y_1), and (x_2, y_2). Notice that this is a right triangle with legs measuring $|x_2 - x_1|$ and $|y_2 - y_1|$. Applying the Pythagorean Theorem, we get the desired result. □

4. Let $y = m_1x + b_1$ and $y = m_2x + b_2$ be two perpendicular lines, with $m_1 \neq m_2$. These two lines will intersect at the point $(\frac{b_2-b_1}{m_1-m_2}, \frac{m_1(b_2-b_1)+b_1(m_1-m_2)}{m_1-m_2})$. We'll now choose two points from each of the lines. To make it simple, we choose $(0, b_1)$ on $y = m_1x + b_1$ and $(0, b_2)$ on $y = m_1x + b_1$.

   We now calculate the distances between each of the points. It can be helpful to first define a few variables. Let $\alpha = b_2 - b_1$ and $\beta = m_1 - m_2$. The distance between $(0, b_1)$ and $(0, b_2)$ is $|b_1 - b_2| = |\alpha|$. The distance from $(0, b_1)$ to the intersection will then be

$$\sqrt{(\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha + b_1\beta}{\beta} - b_1)^2} = \sqrt{(\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha}{\beta})^2}.$$

   The distance from $(0, b_2)$ to the intersection will be

$$\sqrt{(\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha + b_1\beta}{\beta} - b_2)^2} = \sqrt{(\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha - \alpha\beta}{\beta})^2}.$$

   These three line segments form a right triangle, so we use the Pythagorean Theorem:

$$\begin{aligned}
& (\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha}{\beta})^2 + (\frac{\alpha}{\beta})^2 + (\frac{m_1\alpha - \alpha\beta}{\beta})^2 = \alpha^2 \\
\implies & \frac{\alpha^2}{\beta^2}(1 + m_1 + 1 + m_1^2 - 2m_1\beta + \beta^2) = \alpha^2 \\
\implies & \frac{\alpha^2}{\beta^2}(1 + m_1 + 1 + m_1^2 - 2m_1^2 + 2m_1m_2 + m_1^2 - 2m_1m_2 + m_2^2) = \alpha^2 \\
\implies & \frac{\alpha^2}{\beta^2}(2 + m_1^2 + m_2^2) = \alpha^2 \\
\implies & 2 + m_1^2 + m_2^2 = \beta^2 \\
\implies & m_1^2 - 2m_1m_2 + m_2^2 = 2 + m_1^2 + m_2^2 \\
\implies & m_1m_2 = -1 \\
\implies & m_1 = -\frac{1}{m_2},
\end{aligned}$$

   as desired.

   You can also try to prove the reverse on your own: if two lines with slopes m_1 and m_2 satisfy $m_1 = -\frac{1}{m_2}$, then they are perpendicular.

5. *Proof.* The line that passes through (x_1, y_1) and (x_2, y_2) has a slope of $\frac{y_1-y_2}{x_1-x_2}$. Then the equation of the line will be $(y_1 - y) = \frac{y_1-y_2}{x_1-x_2}(x_1 - x)$ in point-slope form.

Plugging in $\frac{y_1+y_2}{2}$ and $\frac{x_1+x_2}{2}$ for x and y gives us

$$\begin{aligned}(y_1 - y) &= \frac{y_1 - y_2}{x_1 - x_2}(x_1 - x)\\ (y_1 - \frac{y_1 + y_2}{2}) &= \frac{y_1 - y_2}{x_1 - x_2}(x_1 - \frac{x_1 + x_2}{2})\\ (y_1 - y_2) &= \frac{y_1 - y_2}{x_1 - x_2}(x_1 - x_2)\\ 1 &= 1\end{aligned}$$

Thus $(\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2})$ is on the line.

We'll now show that $(\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2})$ is equidistant to both of the points. By the Distance Formula, the distance from $(\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2})$ to (x_1, y_1) is

$$\sqrt{(\frac{x_1 + x_2}{2} - x_1)^2 + (\frac{y_1 + y_2}{2} - y_1)^2} = \sqrt{(\frac{x_2 - x_1}{2})^2 + (\frac{y_2 - y_1}{2})^2}$$

Similarly, the distance from $(\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2})$ to (x_2, y_2) is

$$\sqrt{(\frac{x_1 + x_2}{2} - x_2)^2 + (\frac{y_1 + y_2}{2} - y_2)^2} = \sqrt{(\frac{x_1 - x_2}{2})^2 + (\frac{y_1 - y_2}{2})^2}$$

Since $(\frac{x_1-x_2}{2})^2 = (\frac{x_2-x_1}{2})^2$ (and likewise for y_1 and y_2), the two distances are equal. Thus, $(\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2})$ is equidistant to (x_1, y_1) and (x_2, y_2) and is on the line that passes through them, making it the midpoint of the two points. □

6. Since this is a triangle, any ordering of the points will be in clockwise or counter-clockwise order. We list the vertices vertically:

$$\begin{gathered}(2, 2)\\ (5, 4)\\ (4, 1)\\ (2, 2)\end{gathered}$$

After drawing shoelaces from 2 to 4, 5 to 1, and 4 to 2, we get the sum $2 \cdot 4 + 5 \cdot 1 + 4 \cdot 2$. For the other sum, we draw shoelaces from 2 to 5, 4 to 4, and 1 to 2 to get $2 \cdot 5 + 4 \cdot 4 + 1 \cdot 2$. Overall, the Shoelace Theorem gives us an area of

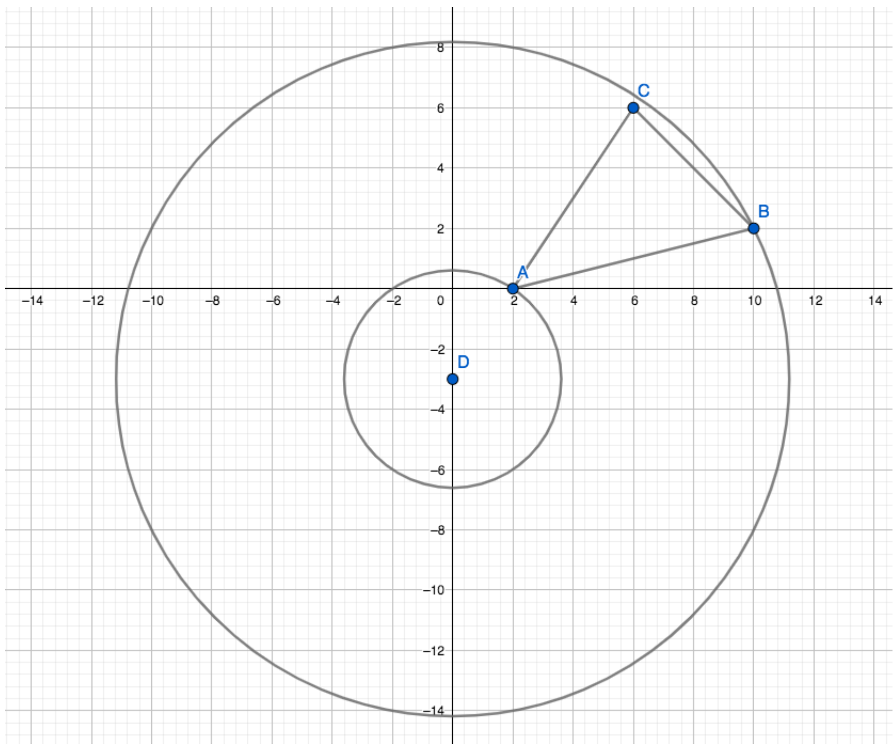
$$\frac{1}{2}|(2 \cdot 4 + 5 \cdot 1 + 4 \cdot 2) - (2 \cdot 5 + 4 \cdot 4 + 1 \cdot 2)| = \boxed{\frac{7}{2}}$$

29.3 Exercises

1. The y-intercept of Chris's line is 7, so the y-intercept of Sebastian's line is 14. Since the x-intercept of Sebastian's line is also double Chris's line, Sebastian's line will have the same slope as Chris's line. Thus, $a + b = 3 + 14 = \boxed{17}$.

2. Notice that (x, y) must lie on the circle with radius 5 centered at $(3, -4)$ by the equation in the problem. Now, to maximize $x^2 + y^2$, we want to (x, y) to be the point on this circle that is the farthest from the origin. It's intuitive that (x, y) must lie on the diameter of the circle through the origin. Thus, the midpoint of the segment with endpoints at the origin and (x, y) is the center of the circle. We have the equation $\frac{x+0}{2} = 3$ and $\frac{y+0}{2} = -4$. Hence $(x, y) = (6, -8)$, so the maximum value of $x^2 + y^2$ is $(6)^2 + (-8)^2 = \boxed{100}$.

3. First, we find the distance between each point of the triangle and the center of rotation. It can help to make a sketch of the diagram:



The distances will be $\sqrt{13}$, $5\sqrt{5}$, and 10. Since $\sqrt{13} < 10 < 5\sqrt{5}$, the point $(6, 6)$ lines inside the annulus. We then have two circles, a big one with area 125π and a smaller one inside with area 13π. The area of the annulus will then be $125\pi - 13\pi = \boxed{112\pi}$.

4. To solve this problem, we will use Pick's formula, $\frac{b}{2} + i - 1 = A$ and solve for i. We are given that the area of the figure is 40, so we have $\frac{b}{2} + i - 1 = 40$. Now we must find b, or the number of point on the border of the figure. To do this, we must first find the coordinates of point C. Given that the figure has a vertical line of symmetry, C must have x-coordinate $\frac{4}{2} = 2$. Let $C = (2, y)$. Splitting the figure into the square $ABDE$ and a triangle BCD, we compute the area of the figure to be $4^2 + \frac{1}{2} \cdot 4 \cdot y - 4 = 40$. Solving, we have $y = 16$. Hence $C = (2, 16)$.

   Now we are ready to find b for the pentagon. First, we will find the number of lattice points on BC. The equation of the line through BC is $y = 6x + 4$.

The number of lattice points between BC (inclusive) can at most be 3, as the x-coordinates can either be 0, 1, or 2. These points are $(0,4)$, $(1,10)$, and $(2,16)$ - all lattice points. Hence there are 3 lattice points on BC. By symmetry, there will also be 3 lattice points on CD. However, the point C has been counted twice - so there is a total of $3+3-1=5$ lattice points on BC and CD.

Now, we will look at the sides of the square - BA, AE, and ED. BA contains 4 lattice points, excluding B and including A. AE contains 4 lattice points, excluding A and including E. Finally, ED contains 3 lattice points, excluding E and D. This gives us a total of $5+4+4+3=16$ lattice points on the border of the figure, so $b=16$. Substituting into Pick's formula, we have $\frac{16}{2}+i-1=40$. Solving for i, we get the number of points on the interior of the figure is $\boxed{33}$.

5. Let $A=(0,0)$. Then $B=(-6,3)$, $C=(4,-1)$, and $D=(2,-3)$. To find the coordinates of point E, we will find the equations of the lines through AB and CD and find their intersection. The line through $(0,0)$ and $(-6,3)$ is $y=-\frac{1}{2}x$. The line through points $(4,-1)$ and $(2,-3)$ is $y=x-5$. So we must find the intersection of the lines $y=-\frac{1}{2}x$ and $y=x-5$. We have $-\frac{1}{2}x=x-5$, so $x=\frac{10}{3}$. Hence $y=\frac{10}{3}-5=-\frac{5}{3}$. So $E=(\frac{10}{3},-\frac{5}{3})$. We have $A=(0,0)$, so using the Distance Formula, we have $AE=\sqrt{(\frac{10}{3}-0)^2+(-\frac{5}{3}-0)^2}=\boxed{\dfrac{5\sqrt{5}}{3}}$

6. We solve this problem using coordinate geometry, even though it can also be solved using area analysis and similarity. We impose the coordinate axes on the square, with D at the origin, A at $(0,s)$, B at (s,s), and C at $(s,0)$, where s is a variable that represents the side length of the square. Note that we want to find s^2. We see that since E is the midpoint of $\overline{CD}$, E is at point $\left(\frac{s}{2},0\right)$. Now, the equation of the line passing through $\overline{DE}$ is $y=2x-s$ and the equation of the line passing through $\overline{AC}$ is $y=-x+s$. By setting these two equations equal to eachother and solving for x and y, we can find the coordinates of point F. Doing the calculations, we have $2x-s=-x+s$ which implies $\frac{2s}{3}$. Plugging this value for x into any of the two original line equations, we obtain that $y=\frac{s}{3}$. Hence, F is at point $\left(\frac{2s}{3}.\frac{s}{3}\right)$. Finally, we use the Shoelace formula on the vertices of quadrilateral $AFED$ starting from vertex D (the origin) and moving clockwise. We have that

$$\begin{aligned}[AFED] &= \frac{\left|0-\left(\frac{2s^2}{3}+\frac{s^2}{6}\right)\right|}{2} \\ &= \frac{5s^2}{12}\end{aligned}$$

Since we were given that the area of $AFED$ is 45, we get that $\frac{5s^2}{12}=45$. Solving for s^2, we have that $s^2=\boxed{108}$.

7. We impose the coordinate axes on rectangle $ABCD$ such that D is at the origin, A is at $(0,3)$, B is at $(5,3)$, and C is at $(5,0)$. Additionally, we have that F is at $(1,0)$ and G is at $(3,0)$. Now, we calculate that the line through points A and F is

given by the equation $y = -3x + 3$ and the line through points B and G is given by the equation $y = \frac{3}{2}x - \frac{9}{2}$. We can set these two equations equal to each other to find the x coordinate of point E. We find $x = \frac{5}{3}$, and by plugging this back into one of the original line equations, we see that point E is located at $\left(\frac{5}{3}, -2\right)$. Now, to compute the area of ΔABE, we use $\overline{AB}$ as the base. The distance from E to side $\overline{AB} = 3 - (-2) = 5$ (we subtract the y coordinate of point E from that of point A). Hence, our desired area is $\frac{5 \cdot 5}{2} = \boxed{\dfrac{25}{2}}$.

8. Let's define A as the origin. Then $E = (4, 0)$, $G = (5, -3)$, $C = (5, -4)$, $F = (2, -4)$, AG is on line $y = -\frac{3}{5}x$, AC is on line $y = -\frac{4}{5}x$, and EF is on line $y = 2x - 8$.

   Q is the intersection of AG and EF, which is $(\frac{40}{13}, -\frac{24}{13})$. P is the intersection of AC and EF, which is $(\frac{20}{7}, -\frac{16}{7})$. By the Distance Formula, $QP = \frac{20\sqrt{5}}{91}$. Also by the Distance Formula, $EF = 2\sqrt{5}$. So the answer is $\frac{PQ}{EF} = \boxed{\dfrac{10}{91}}$.