Summer Math Circle Handouts

July 20, 2019

7 Solutions to Probability and Expected Value Problems

7.1 Warm-up Problems

1. There are three possible odd numbers you can roll out of the six faces of the die, so the probability is $\frac{3}{6} = \boxed{\frac{1}{2}}$.

2. There are three possible ways to roll a sum of 10: $4+6, 5+5, 6+4$. There are $6 \cdot 6 = 36$ ways to roll two die, so the probability will be $\frac{3}{36} = \boxed{\frac{1}{12}}$.

3. For the coin to land heads up exactly twice, we need to choose two flips out of four for the coin to land heads up. There are $\binom{4}{2} = 6$ ways to do this, while there are $2^4 = 16$ outcomes for flipping four coins. Thus, the probability will be $\frac{6}{16} = \boxed{\frac{3}{8}}$.

4. Inscribed means the circle fits "snugly" inside the square. In other words, the circle touches the square at the four midpoints of the square's sides. Let r be radius of the circle. The area of the circle is πr^2, and the area of the square is $(2r)^2 = 4r^2$. Thus, the probability that the point is in the circle will be $\frac{\pi r^2}{4r^2} = \boxed{\frac{\pi}{4}}$.

5. The probability that the three digits form a palindrome is $\frac{10^2}{10^3} = \frac{1}{10}$ since to make a 3 digit palindrome, you must choose a first digit and a middle digit (the last digit is the same as the first). Similarly, the probability that the three lets form a palindrome $\frac{26^2}{26^3} = \frac{1}{26}$. Since these two probabilities are independent, the probability that both the 3 digits and the 3 letters form palindromes is $\frac{1}{10} \cdot \frac{1}{26} = \frac{1}{260}$. Hence, recognizing the overcounting (remember PIE?), our answer is $\frac{1}{10} + \frac{1}{26} - \frac{1}{260} = \boxed{\frac{7}{52}}$.

7.2 Probability Exercises

1. There are two numbers divisible by 3 on a die, and there is a $\frac{1}{2}$ chance of flipping heads. So, the overall probability will be $\frac{2}{6} \cdot \frac{1}{2} = \boxed{\frac{1}{6}}$.

2. The probability that one of the dice rolled is 1 is $\frac{1}{6}$, so the probability that the other dice rolled are *not* 1 is $(\frac{5}{6})^9$. There are also 10 ways to choose which die is 1, so the overall probability will be $\boxed{10 \cdot \frac{5^9}{6^{10}}}$.

3. The general solution is similar to the problem above. There is a p^k probability that k flips come up heads, and a $(1-p)^{n-k}$ probability that the rest of the flips come up tails. In addition, there are $\binom{n}{k}$ ways to choose the k flips that come up heads. Thus, the total probability will be $\binom{n}{k}p^k(1-p)^{n-k}$.

4. First, let's find the number of ways to choose two bills. There are $\binom{4}{2} = 6$ ways to choose two different bills, and 4 ways to choose two of the same bills. In order for the sum to be greater than \$20, we must draw at least one \$20 bill — there are four ways to do so — or draw two \$10 bills. Thus, the probability is $\frac{4+1}{10} = \boxed{\frac{1}{2}}$.

5. The only case where their bags have the same contents is when Bob gives the same color ball that Alice chose back to Alice. It doesn't matter what color Alice picks. When Bob selects a ball, there are 2 balls with the same color that Alice chose out of 6 balls in total. Thus, the probability will be $\frac{2}{6} = \boxed{\frac{1}{3}}$.

6. A, B, C, and D will form a cyclic quadrilateral (basically a quadrilateral with its 4 vertices on the circle). WLOG fix A at one of the vertices. AB and CD intersect if B is opposite of A (so that AB is a diagonal of the quadrilateral). There are 3 places where B can go, so there is a $\boxed{\frac{1}{3}}$ chance that AB is a diagonal, and intersects with CD.

7. For $ab+c$ to be even, ab and c must have the same parity (that means they must both be even, or both be odd). We'll split the problem into two cases:

   Case 1: ab and c are both even. There is a $\frac{2}{5}$ chance that c is even. For ab to be even, at least one of a or b must be even. We can compute the number of ways to choose a and b with complementary counting. Overall, there are $5^2 = 25$ ways to choose two numbers from the set, and there are $3^2 = 9$ ways to choose two odd numbers. So, there are $25 - 9 = 16$ ways to choose two numbers with at least one even number. Each way has a $(\frac{1}{5})^2 = \frac{1}{25}$ probability of occurring, so the probability of ab being even is $\frac{16}{25}$. Thus, there is a $\frac{16}{25} \cdot \frac{2}{5} = \frac{32}{125}$ probability that ab and c are both even.

   Case 2: ab and c are both odd. There is a $\frac{3}{5}$ chance that c is odd. ab can only be even or odd. We know the probability of ab being even is $\frac{16}{25}$, so the probability of ab being odd is $1 - \frac{16}{25} = \frac{9}{25}$. Thus, the probability that ab and c are both odd will be $\frac{3}{5} \cdot \frac{9}{25} = \frac{27}{125}$.

Combining the two probabilities, there is a $\frac{32}{125} + \frac{27}{125} = \boxed{\dfrac{59}{125}}$ probability that $ab + c$ is even.

8. We'll split this into two cases:

   Case 1: Both balls are green. There is a $\frac{4}{4+6} = \frac{2}{5}$ chance that the ball from the first urn is green, and a $\frac{16}{16+N}$ chance that the ball from the second urn is green. The overall probability will be $\frac{2}{5} \cdot \frac{16}{16+N}$.

   Case 2: Both balls are blue. There is a $\frac{6}{4+6} = \frac{3}{5}$ chance that the ball from the first urn is blue, and a $\frac{N}{16+N}$ chance that the ball from the second urn is blue. The overall probability will be $\frac{3}{5} \cdot \frac{N}{16+N}$.

   We add the two probabilities to get the equation $\frac{2}{5} \cdot \frac{16}{16+N} + \frac{3}{5} \cdot \frac{N}{16+N} = \frac{58}{100}$. Now we just grind away at the equation until we get N.

$$\begin{aligned} \frac{2}{5} \cdot \frac{16}{16+N} + \frac{3}{5} \cdot \frac{N}{16+N} &= \frac{58}{100} \\ 2 \cdot 16 + 3 \cdot N &= \frac{29}{50} \cdot 5 \cdot (16 + N) \\ 32 + 3N &= 2.9 \cdot 16 + 2.9N \\ 0.1N &= 2.9 \cdot 16 - 32 \\ N &= \boxed{144} \end{aligned}$$

9. Without loss of generality, let the matching pair removed be 10s. There are now two possibilities:

   Case 1: We choose a matching pair that aren't 10s. There are $\frac{36}{38}$ probability to choose a card that is not 10, and there is a $\frac{3}{37}$ probability to then choose a matching card. The total probability will be $\frac{36}{38} \cdot \frac{3}{37}$.

   Case 2: We choose the remaining two 10s. There is a $\frac{2}{38} \cdot \frac{1}{37}$ probability of doing so.

   We add the two probabilities to get the overall probability of $\frac{36}{38} \cdot \frac{3}{37} + \frac{2}{38} \cdot \frac{1}{37} = \boxed{\dfrac{55}{703}}$.

10. We can write the probability that Amelia wins as a geometric series by computing the probability she wins on the fist turn, the third turn, the fifth turn, and so on. Amelia has a $\frac{1}{3}$ chance of winning on a turn. During the rest of the turns, Amelia and Blaine must flip tails. For example, Amelia has a $\frac{1}{3}$ chance of wining on the first turn, a $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3}$ chance of winning on the third turn, a $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3}$ chance of winning on the fifth term, and so on. Adding all of these probabilities together, we get the series:

$$\frac{1}{3} + (\frac{2}{3} \cdot \frac{3}{5}) \cdot \frac{1}{3} + (\frac{2}{3} \cdot \frac{3}{5})(\cdot \frac{2}{3} \cdot \frac{3}{5}) \cdot \frac{1}{3} + \cdots$$

Use the sum formula, this evaluates to

$$\frac{\frac{1}{3}}{1-\frac{2}{3}\cdot\frac{3}{5}} = \boxed{\frac{5}{9}}$$

.

11. Let's find the number of ways that 7 dice can sum to 10. There are 7 ways to order this partition this:

$$1+1+1+1+1+1+4$$

There are $\frac{7!}{5!} = 42$ ways to order this:

$$1+1+1+1+1+2+3$$

There are $\frac{7!}{4!3!} = 35$ ways to order this:

$$1+1+1+1+2+2+2$$

Each arrangement has a $\frac{1}{6^7}$ chance of occurring, so the total probability will be $\frac{1}{6^7}(7+42+35)$ and $n = \boxed{84}$.

12. First we'll find the probability of rolling each face. Let the probability of rolling a 1 be x. All the probabilities must add to 1, so $x+2x+3x+4x+5x+6x = 1 \implies x = \frac{1}{21}$.

    Now we can find the probability of rolling a sum of 7. There are 6 different combinations:

    $(1,6)$ or $(6,1)$ — there is a $2\cdot\frac{1}{21}\cdot\frac{6}{21}$ probability of these two.

    $(2,5)$ or $(5,2)$ — there is a $2\cdot\frac{2}{21}\cdot\frac{5}{21}$ probability for this case.

    $(3,4)$ or $(4,3)$ — there is a $2\cdot\frac{3}{21}\cdot\frac{4}{21}$ probability for this case.

    Adding up these probabilities, we get $2\cdot\frac{1}{21}\cdot\frac{6}{21}+2\cdot\frac{2}{21}\cdot\frac{5}{21}+2\cdot\frac{3}{21}\cdot\frac{4}{21} = \boxed{\frac{8}{63}}$.

13. We use complementary counting. We'll find the probability that the third shiny penny will appear within the first four draws. There are 2 cases:

    Case 1: The third shiny penny appears on the third draw. This means the first 3 draws are all shiny pennies. There is a $\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5} = \frac{1}{35}$ probability of this occuring.

    Case 2: The third shiny penny appears on the fourth draw. For this to occur, we must draw two shiny pennies and one dull penny during the first three draws. There are 3 ways to order this. No matter what order, the denominator of the probability will be $7\cdot6\cdot5\cdot4$ and the numerator will be $4\cdot3\cdot2\cdot1$, so the overall probability will be $3\cdot\frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1} = \frac{3}{35}$.

    Thus, the overall probability of the third shiny penny appearing in the first four draws will be $\frac{1}{35}+\frac{3}{35} = \frac{4}{35}$. We subtract this from 1 to get the probability that the third penny takes more than four draws: $1-\frac{4}{35} = \boxed{\frac{31}{35}}$.

14. There are $\binom{5}{2} = 10$ ways for Tina two select two distinct numbers. Note that the range of the sum of Tina's two number is 3 to 9, inclusive. We'll take this problem using a case-by-case basis with respect to Sergio's number:

    1: Tina's lowest possible sum is 3, so it's impossible for Sergio's number to be greater if he chooses 1. The probability for 1 is 0.

    2: Also 0.

    3: Also 0.

    4: Sergio's number will be greater if Tina has a sum of 3. $\frac{1}{10}$ probability.

    5: Sergio's number will be greater if Tina has a sum of 3 or 4. $\frac{1}{10} + \frac{1}{10}$ probability.

    6: Same idea as above, sums of 3, 4, or 5 work. $\frac{1}{10} + \frac{1}{10} + \frac{2}{10}$.

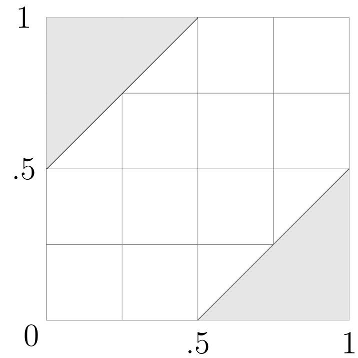
    7: $\frac{1}{10} + \frac{1}{10} + \frac{2}{10} + \frac{2}{10}$.
    8: $\frac{1}{10} + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{2}{10}$
    9: $\frac{1}{10} + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{2}{10} + \frac{1}{10}$
    10: $\frac{1}{10} + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{2}{10} + \frac{1}{10} + \frac{1}{10}$
    Sergio has a $\frac{1}{10}$ probability of selecting any of his numbers, so the overall probability will be $\frac{1}{10} \cdot 16 \cdot \frac{1}{10} \cdot 12 \cdot \frac{2}{10} = \boxed{\frac{2}{5}}$.

7.3 Geometric Probability Review Exercises

1. In order to catch the bus, the bus must arrive between 12:45 and 1:00. This probability is simply $\frac{15}{60} = \boxed{\frac{1}{4}}$.

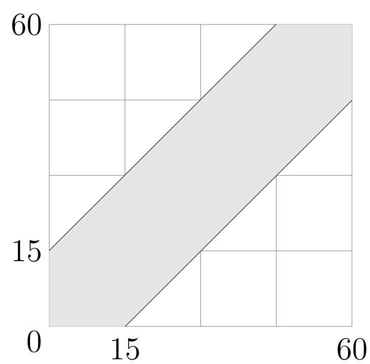
2. Let x be Chloé's number and y be Laurent's number. We can represent this problem on the coordinate plane. The total area of all the outcomes is a 2017×4034 rectangle. The region where Chloé's number is greater is represented by the inequality $x > y$, which is a quarter slice of the rectangle. Thus, the probability that Laurent's number is greater will be $1 - \frac{1}{4} = \boxed{\frac{3}{4}}$.

3. We look at the 1×1 square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ on the coordinate axes. If x and y are the two numbers, we want to find the region of all points in this square such that $|x - y| > \frac{1}{2}$. This region is shown below.



Hence, the area of the desired region is $2 \cdot \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4}$, so our final answer is $\frac{\frac{1}{4}}{1} = \boxed{\frac{1}{4}}$.

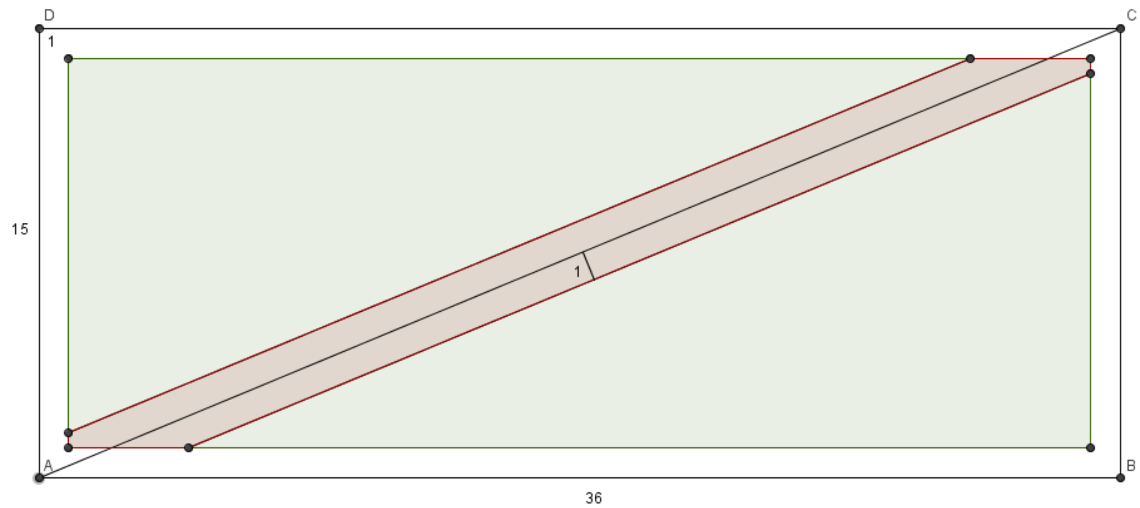
4. We look at the 60×60 square with vertices $(0,0)$, $(60,0)$, $(60,60)$, and $(0,50)$ on the coordinate axes with the unit as minutes. If Allen and Bethany arrive$x$ and y minutes after 1:00, respectively, then the probability that they see each other is represented by the region $|x - y| \leq 15$ in the larger square. This is shown in the figure below.



Using complimentary counting, the area of this region is $60^2 - 2\frac{1}{2} \cdot 45^2 = 1575$. Since the area of the whole square is $60^2 = 3600$, our probability is $\frac{1575}{3600} = \boxed{\frac{7}{16}}$.

5. Notice that this is just a generalization from the previous problem. We look at the 60×60 square with vertices $(0,0)$, $(60,0)$, $(60,60)$, and $(0,50)$ on the coordinate axes with the unit as minutes. If the two mathematicians arrive x and y minutes after 1:00, then the probability that they see each other is represented by the region $|x - y| \leq m$ in the larger square. From the previous problem, the area of this region is simply $60^2 - 2\frac{1}{2} \cdot (60 - m)^2 = 120m - m^2$. Hence, we get the equation $\frac{120m - m^2}{3600} = 40\%$. Rearranging, we get the quadratic $m^2 - 120m + 1440 = 0$. Solving this, m is either $60 - 12\sqrt{15}$ or $60 + 12\sqrt{15}$. Since $m \leq 60$, we have that $m = \boxed{60 - 12\sqrt{15}}$.

6. Since the circle must be completely inside the rectangle, the circle's center must be in the 13×34 rectangle that is contained in and parallel to the original 15×36 rectangle. Now, we find that the region inside this smaller rectangle that contains all the points at least 1 unit away from $\overline{AC}$ is 2 right triangles. These triangles represent the region where the center of the circle can be and are shaded in green in the figure below *(Credit: AoPS)*.



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Now, there are many ways to find the area of each of these two triangles such as similarity, inradius analysis, or even coordinate geometry. Either way, you get that the area of each of the triangles is $\frac{375}{2}$ (we encourage you to actually do the computation and find the area since it is a great way to practice various geometry techniques). Hence, our probability is $\frac{2 \cdot \frac{375}{2}}{13 \cdot 34} = \boxed{\frac{375}{442}}$.

7.4 Conditional Probability Review Exercises

1. In this problem, B is tossing at least one head. We can use complementary counting to calculate $P(B)$. The probability that no heads are tossed is $\frac{1}{32}$, so $P(B) = \frac{31}{32}$.

   $A \cap B$ are the outcomes where we toss at least one head and toss at least three heads; basically tossing at least three heads. There are $\binom{5}{2} = 10$ ways to toss three heads and two tails, 5 ways to toss four heads and one tail, and 1 way to toss five heads. Each way has $(\frac{1}{2})^5$ chance of occurring, so the total probability is $\frac{16}{31}$.

   So,

$$P(A|B) = \frac{\frac{16}{32}}{\frac{31}{32}} = \boxed{\frac{16}{31}}$$

2. This time, event B is tossing at least one tail. The probability of this will be the same as tossing at least one head, so $P(B) = \frac{31}{32}$.

$A \cap B$ are the outcomes where we toss at least one tail and toss at least three heads. There are two cases where this occurs: there are $\binom{5}{2} = 10$ ways to toss three heads and two tails and 5 ways to toss four heads and one tail, for a total of 15 ways. Then, $P(A \cap B) = \frac{15}{32}$.

So,

$$P(A|B) = \frac{\frac{15}{32}}{\frac{31}{32}} = \boxed{\frac{15}{31}}$$

3. Event B is the situation where the combined weight of the first pair selected is equal to the combined weight of the second pair. There are two ways this can occur:

   Case 1: Both pairs consist of two genuine coins. There is a $\frac{8}{10} \cdot \frac{7}{9} \cdot \frac{6}{8} \cdot \frac{5}{7} = \frac{1}{3}$ chance that this occurs.

   Case 2: Both pairs consist of a genuine coin and a counterfeit. There are $\binom{4}{2} = 6$ orders we can draw the coins, but the probability of each order will have numerator $8 \cdot 7 \cdot 2 \cdot 1$ and denominator $10 \cdot 9 \cdot 8 \cdot 7$. This gives us a total probability of $\frac{1}{15}$.

   Thus, $P(B) = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}$. Now, $A \cap B$ is the probability that all four coins are genuine, which already implies that their combined weights are the same. We know the probability of this occurring is $\frac{1}{3}$ from above. So,

$$P(A|B) = \frac{\frac{1}{3}}{\frac{2}{5}} = \boxed{\frac{5}{6}}$$

4. First we find the probability that a man has all of the risk factors. Let this probability be x. Then, the probability that the man has risk factors A and B is $x + 0.14$. Hence, using the fact that the probability a man has all three risk factors, given that he has A and B is $\frac{1}{3}$, we get the equation $\frac{x}{x+.14} = \frac{1}{3}$. Hence, $x = .07$. Next, we need to find the probability that he has none of them. Using a Venn Diagram, this probability is simply $1 - 3 \cdot .1 - 3 \cdot .14 - .07 = .21$. Finally, we need to find the probability that he doesn't have A. This probability is the sum of the probabilities that he has only B (which is .1), he has only C (which is .1), he has only B and C (which is .14), and he has none of them (which is .21). Thus, the probability that he doesn't have A is $.1 + .1 + .14 + .21 = .55$. Therefore, our desired conditional probability is $\frac{.21}{.55} = \boxed{\frac{21}{55}}$.

7.5 Expected Value Review Exercises

1. The expected value will be $\frac{1}{2} \cdot 50 + \frac{1}{2} \cdot (-30) = 10$ (note that this is just the average of the two values). However, you pay \$10 to play the game, so the expected amount of money you gain is $\boxed{\$0}$. This means that the game is fair.

2. From the dice example, we know that the expected value of rolling a die is $\frac{7}{2}$. Thus, the expected sum from rolling two die will be $\frac{7}{2} + \frac{7}{2} = \boxed{7}$.

3. There is a $\frac{1}{1{,}000{,}000}$ chance of winning the lottery. Thus, the expected value will be $\frac{1}{1{,}000{,}000} \cdot 100{,}000{,}000 = \boxed{\$100}$.

4. There is a $\frac{1}{6}$ probability of rolling one 1 on one die, so the expected number of 1s from rolling one die is $\frac{1}{6} \cdot 1 = \frac{1}{6}$. Thus, the expected number of 1s from rolling two dice is $\frac{1}{6} + \frac{1}{6} = \boxed{\frac{2}{3}}$

5. We'll generate a table of the different possible outcomes. The larger number in a dice roll is given by the intersection of the rows and columns.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Each combination has a $\frac{1}{36}$ probability of occuring. As we can see, 6 will be the greater number for 11 of the outcomes, 5 is the greater number for 11, and so on. Thus, the expected value will be $\frac{1}{36}(11 \cdot 6 + 9 \cdot 5 + 7 \cdot 4 + 5 \cdot 3 + 3 \cdot 2 + 1 \cdot 1) = \boxed{\frac{161}{36}}$.

6. Each of the cards can either be on top of the two jokers, in between the jokers, or beneath the jokers. Thus, each card has a $\frac{1}{3}$ probability of being between the jokers. This means the expected number of cards between the two jokers will be $52 \cdot \frac{1}{3} = \boxed{\frac{52}{3}}$.

7. Jim has a $\frac{1}{2}$ chance of flipping heads in 1 flip, a $(\frac{1}{2})^2$ chance of flipping heads in 2 flips, a $(\frac{1}{2})^3$ chance of flipping heads in 3 flips, and so on. The expected value will then be
$$1 \cdot \frac{1}{2} + 2 \cdot (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2})^3 + \cdots$$
This is an **arithmetico–geometric series** (see the chapter on Sequences and Series for more information). To evaluate it, let
$$S = 1 \cdot \frac{1}{2} + 2 \cdot (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2})^3 + \cdots$$
Then
$$\frac{1}{2}S = 1 \cdot (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2})^3 + 2 \cdot (\frac{1}{2})^4 + \cdots$$

If we subtract the second equation from the first, we get

$$\frac{1}{2}S = 1 \cdot \frac{1}{2} + 1 \cdot (\frac{1}{2})^2 + 1 \cdot (\frac{1}{2})^3$$

Now we can just evaluate this using the sum formula to get $\frac{1}{2}S = \frac{\frac{1}{2}}{1-\frac{1}{2}} \implies S = \boxed{2}$.