Summer Math Circle Handouts

July 14, 2019

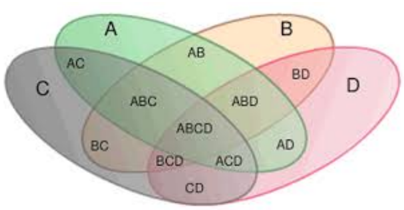
6 Solutions to PIE and Venn Diagram Problems

6.1 Checkpoints

1. We add 331 and 45 to get 376. The number of adults that own both motorcycles and cars is counted twice, so $376 - 351 = 25$ adults own both. As a result, $351 - 25 = 306$ adults own only cars.

2. Assume there are x students taking all three classes. $10 + 13 + 9 = 32$ overcounts the number students taking exactly two courses once and overcounts the number of students taking three courses twice (you can use PIE or a Venn Diagram to visualize this). Thus, we can subtract by 9, the number of students taking at least two courses. However, we are still overcounting the number of students taking exactly three courses once, so we subtract by x again to get the overall number of students. This means $32 - 9 - x = 20 \implies x = \boxed{3}$.

3. Unlike what you might think, drawing four overlapping circles is incorrect because it misses two intersections. This is one example of a correct Venn diagram with 4 sets:



Feel free to use Google to find other examples of Venn diagrams (there are some pretty cool ones).

4. For 4 sets, PIE says:

$$\begin{aligned}|A \cup B \cup C \cup D| = &(|A| + |B| + |C| + |D|) \\ &- (|A \cap B| + |B \cap C| + |C \cap D| + |D \cap A| + |B \cap D| + |A \cap C|) \\ &+ (|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|) \\ &- (|A \cap B \cap C \cap D|).\end{aligned}$$

6.2 Warm-up Problems

1. 10 pepperoni slices and 10 mushroom slices sum to a total of 20 slices. However, the pizza only has 18 slices, so $20 - 18 = \boxed{2}$ of the slices must contain both pepperoni and mushroom.

2. $\frac{3}{4} \cdot 28 = 21$ students in the class have brown hair, and $\frac{6}{7} \cdot 28 = 24$ students are right-handed. $21 + 24 = 45$, so the smallest number of students that could be both is $45 - 28 = \boxed{17}$.

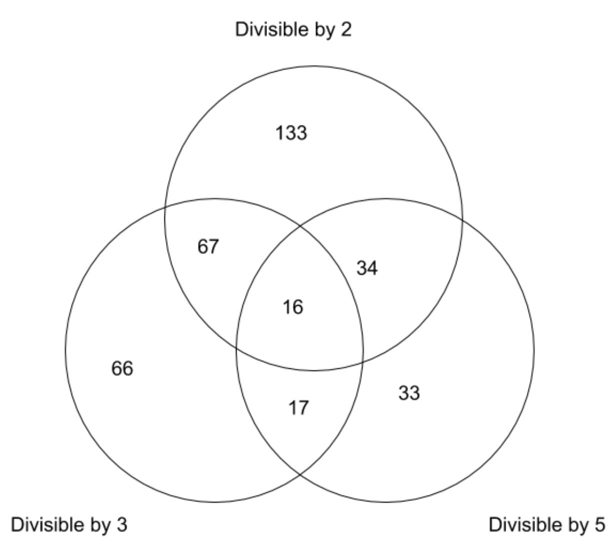
6.3 Basic Set Theory Review Exercises

1. We wish to find the number of elements in A and B combined. $a + b$ counts the number of elements in the intersection of A and B twice, so we subtract by the number of elements in the intersection. This gives us $|A \cup B| = \boxed{a + b - n}$.

2. We're trying the find the number of elements that are in both A and B. As we saw in the previous exercise, $a + b$ equals the number of elements in the union plus the number of elements in the intersection, so we can simply subtract $a + b$ by the number of elements in the union to get the size of the intersection, $\boxed{a + b - u}$.

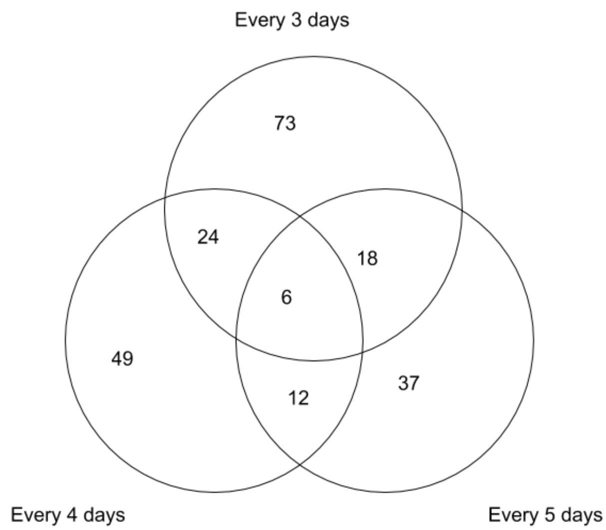
6.4 Venn Diagram Review Exercises

1. We can use complementary counting on this problem by calculating the number of integers that are divisible by 2, 3, **or** 5, then subtracting from the total number of integers. In general, there are $\lfloor \frac{n}{k} \rfloor$ positive integers less than or equal to n that are divisible by k. We can construct a Venn diagram to solve this. By calculating the intersections of the different sets starting from the center, we obtain the diagram below:



We add up all of the numbers in the diagram to find the union of all the sets: $133 + 67 + 16 + 34 + 66 + 17 + 33 = 366$. This means 366 integers less than 500 are divisible by 2, 3, or 5. As a result, $500 - 366 = \boxed{134}$ integers less than 500 are divisible by neither 2, 3, nor 5.

2. Again, we can use complementary counting to solve this problem by finding the number of days where Mrs. Sanders does get a call from any of her grandchildren. We can then use a Venn diagram to find the number of days each grandchildren calls her. Remember that there are $\lfloor \frac{n}{k} \rfloor$ positive integers less than or equal to n that are divisible by k, so in this case $n = 365$ and $k = 3, 4$ and 5. Like before, we start by calculating the intersections and move outwards to obtain the diagram below:



The total number of days where Mrs. Sanders receives a call from her grandchildren will be the union of the three sets: $73+24+6+18+49+12+37 = 219$. Thus, Mrs. Sanders will not receive a call from any of her grandchildren $365-219 = \boxed{146}$ days of the year.

6.5 Problems

1. Let c denote the number of players taking chemistry. Then $8 + c - 4 = 20$, so $c = \boxed{16}$.

2. Let b denote the number of students with blue eyes. Then the number of students with blond hair is $2b$ from the problem statement. Using a 2-set Venn Diagram, we have that $30 - 3 = 2b + b - 6$, where $30 - 3 = 27$ is the number of students with either blond hair or blue eyes (or both). Thus, $b = \boxed{11}$.

3. We do casework on the first three digits. The first digit must be a 1, or otherwise the number won't be 6 digits. The first three digits can either be $(1, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, or $(1, 0, 0)$. The number of valid binary numbers with starting 3 digits $(1, 1, 1)$ is $2^3 = 8$, since the remaining 3 digits can be all be either 0 or 1. The number of valid binary numbers with starting 3 digits $(1, 1, 0)$ is 1, as the only one with 3 consecutive 1's is 110111. The number of valid binary numbers with starting 3 digits $(1, 0, 1)$ is 2, since the only possibilities are 101110 or 101111.

Finally, the number of valid binary numbers with starting 3 digits $(1, 0, 0)$ is 1, since the remaining 3 digits must all be 1. Therefore, our answer is $8+1+2+1 = \boxed{12}$.

4. The probability that the three digits form a palindrome is $\frac{10^2}{10^3} = \frac{1}{10}$ since to make a 3 digit palindrome, you must choose a first digit and a middle digit (the last digit is the same as the first). Similarly, the probability that the three lets form a palindrome $\frac{26^2}{26^3} = \frac{1}{26}$. Since these two probabilities are independent, the probability that both the 3 digits and the 3 letters form palindromes is $\frac{1}{10} \cdot \frac{1}{26} = \frac{1}{260}$. Hence, recognizing the overcounting, our answer is $\frac{1}{10} + \frac{1}{26} - \frac{1}{260} = \boxed{\frac{7}{52}}$.

5. Let's start with a reasonable estimate: 1050. The number of perfect squares less than 1050 is 32. The number of perfect cubes less than 1050 is 10. The number of perfect 5th powers less than 1050 is 4. The number of squares and cubes (equivalent to 6th powers) less than 1050 is 3. The number of squares and 5th powers (equivalent to 10th powers) less than 1050 is 2. The number of 5th powers and cubes (equivalent to 15th powers) less than 1050 is 1. Finally, the only number less than 1050 that is a square, a cube, and a 5th power is 1. Hence, the number of numbers in the list less than or equal to 1050 is $1050-32-10-4+3+2+1-1 = 1009$. Since we want 1000 numbers in the list, our answer is $1050 - 9 = \boxed{1041}$.

6. This is definitely a tedious problem: the PIE expression involves 4 sets. Basically, one set contains the arrangements with the 1st, 2nd, and 3rd people ordered in increasing height. Similarly, the second set contains the arrangements with the 2nd, 3rd, and 4th people ordered in increasing height, the third contains the arrangements with the 3rd, 4th, and 5th people ordered in increasing height, and finally, the 4th set contains the arrangements with the 4th, 5th, and 6th people ordered in increasing height. Let's denote these sets as A, B, C, and D, respectively. Since there are $6! = 720$ total arrangements, our answer is

$$\begin{aligned}
720 &- |A| - |B| - |C| - |D| \\
&+ |A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D| \\
&- |A \cap B \cap C| - |A \cap B \cap D| - |A \cap C \cap D| - |B \cap C \cap D| \\
&+ |A \cap B \cap C \cap D|
\end{aligned}$$

Evaluate each of these individual terms is tedious, but can be done simply using constructive counting. We leave this task to the reader. Once completed, the final answer is $720-120-120-120-120+30+6+20+30+6+30-6-1-1-6+1 = \boxed{349}$.