Iowa City Math Circle Handouts

July 7, 2019

5 Counting, Part 2

5.1 Warm-up Problems

1. A bubble tea shop has 10 different types of tea and 8 different toppings. How many possible orders are there if you pick one tea and 2 different toppings?

2. In how many ways can 7 people sit around a circular table if 3 of these people must sit together? (Two seatings are considered the same if one is a rotation of the other.)

3. This is the same bubble tea shop (10 types of tea and 8 different toppings). If Reece refuses to order matcha tea and lychee jelly topping together, how many possible orders can Reece make (with one tea and 2 different toppings)?

4. How many terms does the expansion of $(x+y)^{100}$ have?

5.2 Binomial Theorem

The Binomial Theorem is a powerful tool for solving a variety of counting and algebra problems. The Binomial Theorem says:

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

We can use a combinatorial argument to prove why the Binomial Theorem is true.

Proof. We can write $(a+b)^n = \underbrace{(a+b)\cdot(a+b)\cdot(a+b)\cdot\dots\cdot(a+b)}_{n}$. Repeatedly using the distributive property, we see that for a term a^mb^{n-m}, we must choose m of the n terms to contribute an a to the term, and then each of the other $n-m$ terms of the product must contribute a b. Thus, the coefficient of a^mb^{n-m} is the number of ways to choose m objects from a set of size n, or $\binom{n}{m}$. Extending this to all possible values of m from 0 to n, we see that $(a+b)^n = \sum_{m=0}^{n}\binom{n}{m}\cdot a^m\cdot b^{n-m}$, as claimed. □

Proof from AoPS Wiki

There are many uses for the Binomial Theorem, first and foremost expanding binomials.

Example 5.1. *Expand $(x+1)^4$ using the Binomial Theorem.*

Solution. Using the Binomial Theorem, we have

$$(x+1)^4 = \binom{4}{0}\cdot x^0\cdot 1^4+\binom{4}{1}\cdot x^1\cdot 1^3+\binom{4}{2}\cdot x^2\cdot 1^2+\binom{4}{3}\cdot x^3\cdot 1^1+\binom{4}{4}\cdot x^4\cdot 1^0 = 1+4x+6x^2+4x^3+x^4$$

△

Checkpoint 5.1. Expand $(2x-1)^4$.

We can also use the Binomial Theorem to find the coefficients of individual terms in binomial expansions.

Example 5.2. *What is the coefficient of the* x^5 *term in* $(2x+1)^7$*?*

Solution. According to the Binomial Theorem, the coefficient of the x^5 term will be $\binom{7}{5}\cdot 2^5\cdot 1^2 = 21\cdot 32 = \boxed{672}$. △

Checkpoint 5.2. Find the coefficient of xy^5 in $(2x+3y)^6$.

One more trick: If you see a polynomial expansion in the form of the Binomial Theorem, you can turn it back into a binomial raised to a power. This may make it easier to evaluate. This will come in handy on many counting and algebra problems.

5.2.1 Review Exercises

1. Compute $1001^3 - 3\cdot 1001^2 + 3\cdot 1001 - 1$.
2. Show that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^n\binom{n}{n} = 0$ using the Binomial Theorem on $(1-1)^n$.
3. Find the hundreds digit of 2011^{2011} using the Binomial Theorem. *Source: 2011 AMC 10B Problem 23*

5.3 Multinomial Coefficients

We can actually generalize the Binomial Theorem to larger polynomials! For instance, if we have the polynomial $(x+y+z)^7$, we see that the coefficient on $x^3y^2z^2$ can be computed by finding the number of distinct arrangements of $xxxyyzz$. Why? Because we get one factor of x or y or z from each factor of $(x+y+z)$. This is computed to be $\frac{7!}{3!2!2!}$, which we conveniently denote $\binom{7}{3,2,2}$, and call a multinomial coefficient.

As a sanity check, you can verify this by expanding something simpler like $(x+y+z)^2$ by both multiplying it out and using multinomial coefficients.

Checkpoint 5.3. What is the coefficient on $x^3y^3z^2$ in the expansion of $(x+y+z)^8$?

5.4 Pascal's Triangle

Pascal's Triangle is a nice way to relate binomial coefficients. To generate it, you can take two adjacent entries, add them, and put the sum underneath the two entries above. The most useful application of Pascal's Triangle is for visualizing nice relationships between binomial coefficients. Below are the first 7 lines of Pascal's Triangle. The values of n indicate the row numbers of Pascal's triangle.

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

5.4.1 Review Exercises

1. How many rows of Pascal's Triangle contain the number 43?
2. Interior numbers begin in the third row of Pascal's Triangle. The sum of the interior numbers in the fourth row is 6. The sum of the interior numbers of the fifth row is 14. What is the sum of the interior numbers of the seventh row?

5.5 Introduction to Combinatorial Identities

In the following sections, we will discuss various combinatorial identities that show up frequently in competition math. The proofs of these identities are equivalently important, however, because the proofs utilize combinatorial and algebraic skills (which can be useful to solve combinatorial problems).

5.5.1 Pascal's Identity

Pascal's Identity is just the relationship in Pascal's Triangle written out! In other words, $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

Checkpoint 5.4. Try to prove this by calculating the number of ways to choose a group of $k+1$ people from $n+1$ people in two different ways. (Hint: Let Bob be a person in the group of $n+1$ people. There are two possibilities - either Bob is in the group of $k+1$ people, or he is not in the group of $k+1$ people.)

If you want to use algebra, try calculating the coefficient on x^{k+1} by computing $(x+1)^{n+1}$ in two different ways.

5.5.2 Hockey Stick Identity

The Hockey Stick Identity probably gets its name from how it looks in Pascal's Triangle - That is, if you move down and to the right adding entries in Pascal's Triangle, the element to the left of where you stopped is what you should get! In summation notation, we're saying

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

5.5.3 Vandermonde's Identity

Vandermonde's Identity says that

$$\sum_{i=0}^{r} \binom{m}{r-i}\binom{n}{i} = \binom{m+n}{r}.$$

A counting argument for this theorem works really well for proving it; additionally, you can use the Binomial Theorem to prove this theorem.

Checkpoint 5.5. Prove Vandermonde's Identity using the Binomial Theorem. (Hint: calculate the coefficient of x^r in $(x+1)^m(x+1)^n = (x+1)^{m+n}$ in two different ways).

5.6 Problems

1. Find the sum of all integers k such that $\binom{23}{4} + \binom{23}{5} = \binom{24}{k}$.
2. If $x + \frac{1}{x} = -5$, what is $x^3 + \frac{1}{x^3}$?
3. Show that $\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$ using algebra and using a counting argument.
4. If a and b are real numbers with $a - b = 1$, what is the least possible value of $a^5 - b^5$?
5. Prove the Binomial Theorem using induction.
6. * Consider all 1000-element subsets of the set 1, 2, 3, ... , 2015. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$. *Source: 2015 AIME I Problem 12*
7. * Given that $\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!}$ find the greatest integer that is less than $\frac{N}{100}$. *Source: 2000 AIME II Problem 7*
8. * The expression
$$(x+y+z)^{2006} + (x-y-z)^{2006}$$
is simplified by expanding it and combining like terms. How many terms are in the simplified expression? *Source: 2006 AMC 12A Problem 24*