Summer Math Circle Handouts

July 7, 2019

5 Solutions to Counting, Part 2 Problems

5.1 Checkpoints

1. Using the Binomial Theorem, $(2x-1)^4$ is

$$\begin{aligned}&\binom{4}{4}(2x)^4+\binom{4}{3}(2x)^3(-1)+\binom{4}{2}(2x)^2(-1)^2+\binom{4}{1}(2x)^1(-1)^3+\binom{4}{0}(-1)^4\\=&16x^4-32x^3+24x^2-8x^1+1\end{aligned}$$

2. According to the Binomial Theorem, the xy^5 term is going to be $\binom{6}{1}(2x)(3y)^5 = 2916xy^5$, so the coefficient is $\boxed{2916}$.

3. The coefficient of $x^3y^3z^2$ will be the multinomial coefficient $\binom{8}{3,3,2} = \frac{8!}{3!3!2!} = \boxed{560}$.

4. *Proof.* Here is the combinatorial proof: There are $\binom{n+1}{k+1}$ ways to choose a group of $k+1$ people from $n+1$ people total.

   Now, let Bob be in the group of $n+1$ people. When choosing the group of $k+1$ people, Bob can either belong in the $k+1$ people or not be in the $k+1$ people. There are $\binom{n}{k}$ ways to choose $k+1$ people that include Bob, and $\binom{n}{k+1}$ ways to choose $k+1$ people that don't include Bob. In total, there are $\binom{n}{k}+\binom{n}{k+1}$ ways to choose the group of $k+1$ people.

   Because in both methods, we are still choosing $k+1$ people, $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$, as desired. $\square$

5. *Proof.* First we'll consider the LHS of $(x+1)^m(x+1)^n = (x+1)^{m+n}$. We can form the x^r term by multiplying the x^i terms from the expansion of $(x+1)^n$ with the x^{r-i} terms from the expansion of $(x+1)^m$. Thus, the coefficient of the x^r term will be $\sum_{i=0}^{r}\binom{m}{r-i}\binom{n}{i}$.

   Alternatively, if we consider the RHS the coefficient of the x^r term is just $\binom{m+n}{r}$ according to the Binomial Theorem. This means that $\sum_{i=0}^{r}\binom{m}{r-i}\binom{n}{i} = \binom{m+n}{r}$, and we are done. $\square$

5.2 Warm-up Problems

1. There are $10\cdot\binom{8}{2} = \boxed{280}$ possible orders.

2. Pretend that the 3 people who must sit together are one person. There are then $(5-1)! = 24$ ways to seat the "5" people around the table. However, there are also $3! = 6$ ways to order the three people sitting together for a total of $6\cdot 24 = \boxed{144}$ possible seatings.

3. If Reece cannot order matcha tea and lychee jelly topping together, there are 7 orders he cannot make. We know from the first problem that there are 280 possible orders, so Reece can only make $280 - 7 = \boxed{273}$ orders.

4. Notice that $(x+y)$ has two terms, $(x+y)^2$ has three terms, and $(x+y)^3$ has four terms, so $(x+y)^{100}$ should have $\boxed{101}$ terms. It should become more clear why after learning about the Binomial Theorem.

5.3 Binomial Theorem Review Exercises

1. $1001^3 - 3 \cdot 1001^2 + 3 \cdot 1001 - 1$ is the expansion of $(1001-1)^3$, which equals $1000^3 = \boxed{1{,}000{,}000{,}000.}$

2. *Proof.* From the Binomial Theorem, it's clear that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots + (-1)^n \binom{n}{n}$ is the expansion of $(1-1)^n$, which equals $0^n = 0$. $\square$

3. We wish to find the hundreds digit of 2011^{2011}. To do this, we take the expression mod 1000. $2011^{2011} \equiv 11^{2011} \pmod{1000}$.

   We can now use the Binomial Theorem to simplify the calculations (remember we are in mod 1000).

$$\begin{aligned} 11^{2011} &\equiv (1+10)^{2011} \\ &\equiv \binom{2011}{0} \cdot 1 + \binom{2011}{1} \cdot 10 + \binom{2011}{2} \cdot 100 \\ &\equiv 1 + 110 + 500 \\ &\equiv 661 \end{aligned}$$

   This is because any term in the expansion with a factor of 1000 can be canceled out. This gives us the hundreds digit $\boxed{6}$.

5.4 Pascal's Triangle Review Exercises

1. Only $\boxed{1}$ row will contain the number 43, the row $n = 43$.

2. One property of Pascal's Triangle is that the sum of elements in row n will add to 2^n. In this problem, the first row is row $n = 0$, so the seventh row is $n = 6$. The sum of the interior elements on row $n = 6$ will be $2^6 - 2 = \boxed{62}$ (we subtract the two 1s on the exterior).

5.5 Problems

1. From Pascal's Identity, we know that $\binom{23}{4} + \binom{23}{5} = \binom{24}{5}$, so k can equal 5. $\binom{24}{5} = \binom{24}{19}$, so k can also equal 19. The sum of all possibilities is then $5 + 19 = \boxed{24.}$

2. If we cube $x+\frac{1}{x}=-5$, we get $x^3+3x+\frac{3}{x}+\frac{1}{x^3}=-125$. Note that the two terms in the middle are equal to $3(x+\frac{1}{x})=-15$, so $x^3-15+\frac{1}{x^3}=-125 \implies x^3+\frac{1}{x^3}=$ $\boxed{-110.}$

3. Algebra argument:

   Proof. From the Binomial theorem, the coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. In addition, we can write $(1+x)^{2n}=(1+x)^n\cdot(1+x)^n$. We claim that the coefficient of x^n in $(1+x)^n\cdot(1+x)^n$ is $\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2$. To see this, any term in the first $(1+x)^n$ of the form x^k must be multiplied with the x^{n-k} term in the second $(1+x)^n$ to give rise to a x^n term in the final expansion. Since the coefficient of x^k in $(1+x)^n$ is $\binom{n}{k}$ and the coefficient of x^{n-k} in $(1+x)^n$ is $\binom{n}{n-k}$, we have that the coefficient of x^{2n} in $(1+x)^n\cdot(1+x)^n$ is $\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}$. Using the basic fact that $\binom{n}{m}=\binom{n}{n-m}$, this expression is equivalent to $\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2$. Hence, we have that $\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$, so we are done. □

   Note that this is a case of Vandermonde's Identity where $m=n$.

   Counting argument:

   Proof. We wish to choose a group of n people from a larger group of $2n$ people. There are $\binom{2n}{n}$ ways to do so.

   Alternatively, consider a subgroup, which we will denote as A, of n people out of the $2n$ people. In picking the n people, 0 of them could be from A and n of them could be from the n remaining people. In this case, there are $\binom{n}{0}\cdot\binom{n}{n}$, or equivalently $\binom{n}{0}\cdot\binom{n}{0}=\binom{n}{0}^2$.

   Additionally, in picking the n people, 1 of them could be from A and $n-1$ of them could be from the n remaining people. In this case, there are $\binom{n}{1}\cdot\binom{n}{n-1}$, or equivalently $\binom{n}{1}\cdot\binom{n}{1}=\binom{n}{1}^2$.

   We can continue in this fashion to get $\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2$ ways to pick the group of n people with respect to A. This gives this our result: $\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$, and we are done. □

4. Using algebraic manipulation and the Binomial Theorem, we get

$$\begin{aligned}
a^5-b^5 &= (a-b)\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)\\
&= (a-b)\left((a-b)^4+5a^3b-5a^2b^2+5ab^3\right)\\
&= (a-b)\left((a-b)^4+5ab\left(a^2-ab+b^2\right)\right)\\
&= (a-b)\left((a-b)^4+5ab\left((a-b)^2+ab\right)\right).
\end{aligned}$$

   Substituting $a-b=1$ into the above expression, we have that $a^5-b^5=1+5ab(1+ab)=5(ab)^2+5ab+1$. Notice that this is a quadratic, so its minimum is at its vertex. This occurs when $ab = \frac{-5}{2\cdot 5} = -\frac{1}{2}$. Plugging $ab = -\frac{1}{2}$ in, we see that the minimum value of $a^5 - b^5$ is $\boxed{-\frac{1}{4}}$.

5. *Proof.* For $n = 1$,

$$(x+y)^1 = x + y = \binom{1}{0}x^{1-0}y^0 + \binom{1}{1}x^{1-1}y^1 = \sum_{k=0}^{1}\binom{1}{k}x^{1-k}y^k$$

Suppose

$$(x+y)^{n-1} = \sum n - 1_{k=0}\binom{n-1}{k}x^{(n-1)-k}y^k$$

Consider $(x+y)^n$.

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\left[\sum_{k=0}^{n-1}\binom{n-1}{k}x^{(n-1)-k}y^k\right] \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{j=0}^{n-1}\binom{n-1}{j}x^{(n-1)-j}y^{j+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{j=0}^{n-1}\binom{n-1}{(j+1)-1}x^{n-(j+1)}y^{j+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k=1}^{n}\binom{n-1}{k-1}x^{n-k}y^k \\
&= \sum_{k=0}^{n}\left[\binom{n-1}{k}x^{n-k}y^k\right] - \binom{n-1}{n}x^0y^n \\
&\quad + \sum_{k=0}^{n}\left[\binom{n-1}{k-1}x^{n-k}y^k\right] - \binom{n-1}{-1}x^ny^0 \\
&= \sum_{k=0}^{n}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]x^{n-k}y^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k
\end{aligned}$$

and the theorem is proved! □

Credit: Harvey Mudd College Mathematics

6. Let M be the desired mean. Then because $\binom{2015}{1000}$ subsets have 1000 elements

and $\binom{2015-i}{999}$ have i as their least element,

$$\begin{aligned}\binom{2015}{1000}M &= 1\cdot\binom{2014}{999}+2\cdot\binom{2013}{999}+\cdots+1016\cdot\binom{999}{999}\\ &= \binom{2014}{999}+\binom{2013}{999}+\cdots+\binom{999}{999}\\ &+\binom{2013}{999}+\binom{2012}{999}+\cdots+\binom{999}{999}\\ &\cdots\\ &+\binom{999}{999}\\ &= \binom{2015}{1000}+\binom{2014}{1000}+\cdots+\binom{1000}{1000}\\ &= \binom{2016}{1001}.\end{aligned}$$

Using the definition of binomial coefficient and the identity $n! = n\cdot(n-1)!$, we deduce that

$$M = \frac{2016}{1001} = \frac{288}{143}.$$

The answer is $\boxed{431}$. *Credit: AoPS Wiki*

7. Multiplying both sides by 19! yields:

$$\frac{19!}{2!17!}+\frac{19!}{3!16!}+\frac{19!}{4!15!}+\frac{19!}{5!14!}+\frac{19!}{6!13!}+\frac{19!}{7!12!}+\frac{19!}{8!11!}+\frac{19!}{9!10!}=\frac{19!N}{1!18!}.$$

$$\binom{19}{2}+\binom{19}{3}+\binom{19}{4}+\binom{19}{5}+\binom{19}{6}+\binom{19}{7}+\binom{19}{8}+\binom{19}{9}=19N.$$

Recall the Combinatorial Identity $2^{19} = \sum_{n=0}^{19}\binom{19}{n}$. Since $\binom{19}{n} = \binom{19}{19-n}$, it follows that $\sum_{n=0}^{9}\binom{19}{n} = \frac{2^{19}}{2} = 2^{18}$.

Thus, $19N = 2^{18} - \binom{19}{1} - \binom{19}{0} = 2^{18} - 19 - 1 = (2^9)^2 - 20 = (512)^2 - 20 = 262124$.

So, $N = \frac{262124}{19} = 13796$ and $\left\lfloor \frac{N}{100} \right\rfloor = \boxed{137}$. *Credit: AoPS Wiki*

8. By the Multinomial Theorem, the summands can be written as

$$\sum_{a+b+c=2006} \frac{2006!}{a!b!c!}x^a y^b z^c$$

and

$$\sum_{a+b+c=2006} \frac{2006!}{a!b!c!}x^a (-y)^b (-z)^c,$$

respectively. Since the coefficients of like terms are the same in each expression, each like term either cancel one another out or the coefficient doubles. In each expansion there are:

$$\binom{2006+2}{2} = 2015028$$

terms without cancellation. For any term in the second expansion to be negative, the parity of the exponents of y and z must be opposite. Now we find a pattern:

if the exponent of y is 1, the exponent of z can be all even integers up to 2004, so there are 1003 terms.

if the exponent of y is 3, the exponent of z can go up to 2002, so there are 1002 terms.

$\vdots$

if the exponent of y is 2005, then z can only be 0, so there is 1 term.

If we add them up, we get $\frac{1003 \cdot 1004}{2}$ terms. However, we can switch the exponents of y and z and these terms will still have a negative sign. So there are a total of $1003 \cdot 1004$ negative terms.

By subtracting this number from 2015028, we obtain 1008016 as our answer. *Credit: AoPS Wiki*