Iowa City Math Circle Handouts

June 30, 2019

4 Counting, Part 1

4.1 Warm-up Problems

1. Calculate $\frac{100!}{98!}$.

2. How many ways are there to seat three people in three different chairs?

3. How many ways are there to seat three people in three different chairs around a table, given rotations are the same?

4.2 Constructive Counting

Many math competition problems will ask you to count a certain quantity. Constructive counting collects several pieces of the quantity and combines them using addition or multiplication to get the answer. Several counting concepts use the idea of constructive counting, such as permutations and combinations.

4.2.1 Permutations

Definition. A permutation is a rearrangement of the elements of a set in which order matters. It is very helpful to be able to compute the number of permutations of a set, which, like above, can be reasoned to be $n!$ if you have n distinct elements in the set. Similarly, the number of ordered subsets with k elements is $n(n-1)\cdots(n-k+1)$, or $\frac{n!}{(n-k)!}$.

For example, arranging books on a bookshelf is a permutation. Suppose you have five different books on a bookshelf. How many different ways are there to arrange them? There are 5 possible places to place the first book. Only 4 places are remaining after the first book is placed, so there are 4 possible places to put the second book. Then there are 3 possibilities for the third book, 2 for the fourth book, and 1 for the last book. In total, there are $5! = \boxed{120}$.

Checkpoint 4.1. An ice cream shop has 5 different flavors of ice cream. How many ways are there to order an ice cream cone with 3 scoops, assuming each scoop is a different flavor?

4.2.2 Combinations

Definition. A combination is the selection of elements from a set where order does not matter. Problems often ask you to find the number of combinations of a certain number of elements from a set. The number of ways to choose k elements from a set of n distinct elements is equal to $\frac{n!}{(n-k)!k!}$. We will often express this with binomial coefficient notation: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Suppose you want to choose 3 books out of the five books on the bookshelf to read (again, all books are reading). Using the formula, we get $\binom{5}{3} = \frac{5!}{(5-3)!3!}$. Why does this formula work? There are 5 ways to pick the first book, 4 ways to pick the second, and 3 ways to pick the third for a total of $5 \cdot 4 \cdot 3$. Note that this equal to $\frac{5!}{(5-3)!}$.

However, this does not take into account the order of the books. In one case, we may first take out a Harry Potter book, then a poetry book, then a book of math puzzles. If we instead take out the math puzzle book, then the poetry book, and finally the Harry Potter book, we still have the same set of books. So, we must divide $5 \cdot 4 \cdot 3$ by the number of ways to order the books. This is a permutation of 3 books, so there will be $3! = 6$ ways to rearrange them. Overall, the number of combinations of 3 will be equal to $\frac{5!}{(5-3)!3!} = 10$, which agrees with our formula.

Checkpoint 4.2. A ramen restaurant offers 4 different types of noodles, 2 types of broths, and 7 different toppings. How many ways are there to order a bowl of ramen with 2 types of noodles, 1 type of broth, and 3 different toppings?

4.2.3 Symmetry and Rotation

Sometimes problems will ask you to consider the number of ways to arrange items that can be rotated. For example, **Warm-up Problem 3** asks us to find the number of ways to seat three people in three different chairs around a table, given rotations are the same. An easy way to tackle these types of problems is to fix the location of one item, then find the number of ways to arrange the rest of the elements. For **Warm-up Problem 3**, we might fix the location of the first person at the top of the table. Then there are $2! = 2$ ways to arrange the remaining two people.

In general, there are $(n-1)!$ ways to arrange a set of elements where rotations do not matter.

Checkpoint 4.3. A Senate committee has 5 Democrats, 5 Republicans, and 1 Independent. In how many ways can they sit around a circular table if all the members of each party all sit next to each other? (Two seatings are considered equivalent if one is a rotation of the other.) *Source: AoPS*

Sometimes the symmetry of the rearrangements does not matter. In that case, you will need to divide by the number of possible reflections.

Checkpoint 4.4. In how many ways can you arrange 5 distinct beads on a ring? (Note that two arrangements are the same if one can be rotated or reflected to produce the other.)

You will often need to combine combinations, permutations, and other concepts to solve counting problems.

4.2.4 Review Exercises

1. How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.) *Source: AMC 10*

2. In how many ways can 6 people be seated around a triangular table with 2 people on a side? (Two configurations are considered equivalent if one is a rotation of another.)

3. A bug is at $(0,0)$ on the Cartesian plane. How many paths can the bug take to travel to the point $(3,4)$ if it can only travel up 1 unit or to the right 1 unit?

4.3 Complementary Counting

Complementary counting is often used when the items we are counting must satisfy a certain condition. This technique is useful when solving the "opposite" problem is easier (usually, it is much easier). The general outline for this type of solution is to find the total number of options (when you remove the condition from the problem), and then subtract the answer to the opposite problem from this number.

Example 12. *How many integers from* 1 *to* 18*, inclusive, are not divisible by* 3*?*

Solution. You could do this manually - running through all the integers 1 through 18 and checking if which ones are not divisible by 3. However, there is a faster and easier method using complementary counting. To use complementary counting, we first must solve the opposite problem: How many integers from 1 to 18, inclusive, are divisible by 3? This is an easier problem - we just need to find how many integers are in the sequence $3, 6, \dots 18$, which is $\frac{18}{3} = 6$. So we have solved the opposite problem; now we must see how we can use this answer to solve the original problem. The number of integers that are divisible by 3 and the number of integers that are not divisible by 3 must add up to 18. (Why?) So to solve the original problem, we can just take $18 - 6 = \boxed{12}$. △

Example 13. *How many words with* 3 *letters have at least one "A"? (In this problem, a "word" is just any sequence of letters).*

Solution. If the words has at least one "A," then it can have either 1, 2, or 3 A's. This seems like a lot of cases - we can try complementary counting here. What is the opposite of "at least one?" Thinking mathematically, the opposite of ≥ 1 is < 1, which means the only case we have to consider is 0 A's! So we must find the number of 3-letter words with no A's in it. For each letter, we have 25 options - every letter except A. Hence, the total number of 3-letter words with no A is 25^3. Now to solve the original problem, we have to find the total number of words; this is clearly 26^3. Hence, our final answer is $26^3 - 25^3 = \boxed{1951}$. △

Checkpoint 4.5. How many words with at most 3 letters have at least one "A"?

Checkpoint 4.6. How many 5-digit numbers have at least one zero?

4.3.1 Review Exercises

1. In how many ways can we seat 8 people around a table if two of these people refuse to sit next to each other? (Two seating arrangements are the same if one is a rotation of the other.)

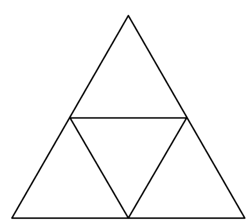
2. Our math club has 20 members, among which we have 3 officers: President, Vice President, and Treasurer. However, one member, Ali, hates another member, Brenda. How many ways can we fill the offices if Ali refuses to serve as an officer if Brenda is also an officer? (No person is allowed to hold more than one office.)

4.4 Problems

1. How many positive integers less than or equal to 100 have a prime factor that is greater than 2?

2. Reece made a list of every possible distinct five-digit positive integer that can be formed using each of the digits 1, 2, 3, 4, and 5 exactly once in each integer. What is the sum of the integers on Reece's list?

3. There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles as shown below. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color. Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles can be constructed? *Source: AoPS*



4. Regular octagon $ABCDEFGH$ has its center at J. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines AJE, BJF, CJG, and DJH are all equal. In how many ways can this be done? *Source: AMC 10*