Summer Math Circle Handouts

June 30, 2019

4 Solutions to Counting, Part 1 Problems

4.1 Checkpoints

1. We are trying to find the number of permutations here, because the order of the ice cream scoops in the cone matter. There are 5 possibilities for the first scoop, 4 for the second, and 3 for the third, giving us a total of $5 \cdot 4 \cdot 3 = \boxed{60}$ possible orders.

2. We need to choose 2 noodles out of 4, a total of $\binom{4}{2}$. We need to choose 1 broth out of 2, or $\binom{2}{1}$. Finally, we need to choose 3 toppings out of 7, for a total of $\binom{7}{3}$. In total, this is $\binom{4}{2} \cdot \binom{2}{1} \cdot \binom{7}{3} = 6 \cdot 2 \cdot 35 = \boxed{420}$ possibilities.

3. All the Democrats need to sit together, so let's just pretend they are one person. All the Republicans also need to sit together, so again let's pretend they are one person. Our problem now asks for the number of ways to seat 3 people around a table, which we know is $(3-1)! = 2$. However, the order of the Democrats and the Republicans sitting around the table matters. There are 5 Democrats and 5 Republicans, so there are 5! ways to order the Democrats and 5! ways for the Republicans. In total, the number of possible seatings will be $2 \cdot 5! \cdot 5! = \boxed{28800}$.

4. First let's ignore reflections of the ring. There are $(5-1)! = 24$ ways to arrange 5 beads around a ring. However, for each arrangement, a symmetrical arrangement can be achieved by flipping the ring over. Thus, to account for reflections we must divide by 2. This gives us $24/2 = \boxed{12}$ arrangements.

5. We'll split this into three cases: 1-letter, 2-letter, and 3-letter words. In each case, we can compute the total number of words and subtract the number of words that contain no As. There are 26 one-letter words and 25 words that contain no As. There are $26 \cdot 26$ 2-letter words, and $25 \cdot 25$ 2-letter words with no As. There are $26 \cdot 26 \cdot 26$ 3-letter words and $25 \cdot 25 \cdot 25$ 3-letter words with no As. This means there are $26^3 - 25^3 + 26^2 - 25^2 + 26 - 25 = \boxed{2003}$ words with at most 3 letters that contain at least one A (to aid in calculations, use the difference of squares and difference of cubes identities).

6. We can count the number of 5-digit numbers that do not contain any 0s and subtract from the total number of 5-digit numbers. There are $9 \cdot 10^4 = 90000$ 5-digit numbers, and 9^5 5-digit numbers that do not contain any 0s. Thus, there are $90000 - 9^5 = \boxed{30951}$ 5-digit numbers that contain at least one 0.

4.2 Warm-up Problems

1.
$$\frac{100!}{98!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdots 1}{98 \cdot 97 \cdots 1} = 100 \cdot 99 = \boxed{9900}$$

   Remember this trick because it is very useful for evaluating combinations and permutations.

2. There are $3! = \boxed{6}$ possible ways (3 places for first person, 2 for second person, and 1 for last person).

3. There are two ways: If A, B, and C are the people, then the two possible ways are ABC and ACB. Later we'll discover that is also equal to $(3-1)! = 2$.

4.3 Constructive Counting Review Exercises

1. Let's number the periods $1, 2, \ldots, 6$. Given the stipulations presented in the problem, the three mathematics courses can be scheduled in the following sets of periods: $\{1,3,5\}, \{1,3,6\}, \{1,4,6\}$ and $\{2,4,6\}$. For each of these 4 sets of periods, there are $3! = 6$ ways to assign each of the three math periods to a mathematics course. Hence, there are a total of $4 \times 6 = \boxed{24}$ ways to schedule the mathematics courses.

2. There are $6! = 720$ ways to seat 6 people in a straight line. However, since we are seating them around a triangular table, we can rotate a given arrangement 3 times to produce an equivalent arrangement. Therefore, the number of ways to seat 6 people around a triangular people with 2 people per side is $\frac{720}{3} = \boxed{240}$.

3. All paths contain 7 moves. Out of the 7 moves, the bug must choose to go to the right three times, so there are $\binom{7}{3} = \boxed{35}$ possible paths.

4.4 Complementary Counting Review Exercises

1. We use complimentary counting. The number of ways to seat 8 people around a table is $(8-1)! = 5040$. In addition, the number of ways to seat 8 people around a table given that two of these people must sit next to each other is $2 \cdot (7-1)! = 1440$ (we group these two people to form a larger person, and there are 2 ways to order these two people within their grouping). Hence, the answer to the problem is $5040 - 1440 = \boxed{3600}$.

2. We use complimentary counting. The number of ways to fill the offices regardless of the stipulation is $3! \times \binom{20}{3} = 6840$. Now, we count the number of ways to fill the offices given that both Ali and Brenda serve. If they both serve, then there is $\binom{18}{1} = 18$ ways to choose the other member who serves, so there are $3! \times 18 = 108$ ways to fill the offices if they both serve. Hence, the answer to the problem is $6840 - 108 = \boxed{6732}$.

4.5 Problems

1. We can calculate the number of integers that do not have a prime factor greater than 2; in other words, integers that are only divisible by 2. There are 7 of these integers: $2^0, 2^1, \cdots, 2^6$. Thus $100 - 7 = \boxed{93}$ integers will have prime factors greater than 2.

2. We will consider the sum digit by digit. There are 24 five-digit integers that end in 5 (let 5 be the ones digit. Then there are $4! = 24$ ways to order the rest of the digits). Likewise, 24 five-digit integers end in 4, 24 end in 3, and so on. Thus the sum of the ones digits will be $24 \cdot 5 + 24 \cdot 4 + \cdots + 24 \cdot 1 = 24(1+2+3+4+5)$. Similarly, the sum of the tens digits will be $24(10+20+30+40+50) = 24 \cdot 10 \cdot (1+2+3+4+5)$, and so on for the remaining digits. Thus the overall sum of all the numbers will be $24(1+2+3+4+5)(1+10+100+1000+10000) = \boxed{3{,}999{,}960}$.

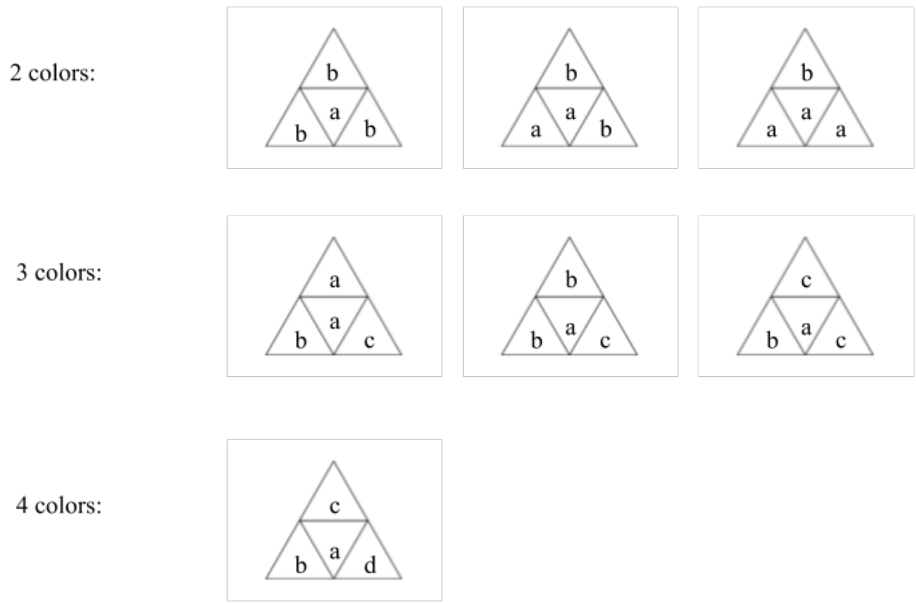
3. We will split this into cases: 1 color, 2 colors, 3 colors, and 4 colors.

   1 color: It is only possible to make a triangle with one solid color. As there are 6 colors, there are 6 possible triangles.

   2 colors: First we need to choose 2 colors out of the 6. There are $\binom{6}{2} = 15$ ways to do this. There are then 3 possible ways to arrangements for the colors, and two ways to further choose how the colors correspond to a and b in the arrangement, according to the figure below, for a total of $15 \cdot 3 \cdot 2 = 90$ triangles.

   3 colors: There are $\binom{6}{3}$ ways to choose three colors. There are 3! ways to "order" the colors among a, b, and c, and there are 3 arrangements for a total of $20 \cdot 6 \cdot 3 = 360$ triangles.

   4 colors: There are $\binom{6}{4} = 15$ ways to choose four colors. There is only one possible arrangement, but there are 4! possible orderings of the colors, which gives us a total of $15 \cdot 1 \cdot 24 = 360$ possible triangles.



Letters in the figure above represent colors.

Thus we have a total of $6 + 90 + 360 + 360 = \boxed{821}$ possible triangles.

4. Let a, b, c, and so on represent the values of the points A, B, C, and so on respectively. Then $a + j + e = b + j + f = c + j + g = d + j + h$. Since each of the points corresponds to one digit from 1 to 9, we know that $(a + j + e) + (b + j + f) + (c + j + g) + (d + j + h) = 45 + 3j$. Furthermore, since all variables are integers, 4 must divide $45 + 3j$. Thus, j can only be 1, 5, or 9.

   We now consider the orderings of the numbers. It suffices to only consider 4 of the points, since giving one point a value will also give its corresponding point a value. Since E is across from A, F is across from B and so on, we will consider A, B, C, and D (we pick these points so that we avoid giving a value to two points that are across from each other, which would be incorrect). There are 8 possible values, however, each of the values is paired to another. This means we are trying to find the number of ways to order these 4 pairs, which is 4!. In addition, there are two options for each point in each pair, so are 2^4 ways to pick the specific numbers to assign to A, B, C, and D.

   The total number of ways will be $3 \cdot 4! \cdot 2^4 = \boxed{1152}$.