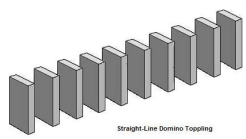
Summer Math Circle Handouts

June 9, 2019

1 Induction

1.1 Domino Exercise

Imagine you have lined up a row of dominoes:



When you push over the first domino, how do you know the domino at the end of the row will be knocked over?

1.2 What is induction?

We know the last domino in the row will fall because the domino before it falls, the domino before the domino before the last domino falls, and so on. In other words, if any domino in the row falls, it will knock down the next domino. Since we know that the first domino in the row falls, then all dominoes in the row will be knocked over. This type of reasoning is the foundation of the mathematical concept called **induction**.

Induction is a powerful proof technique that helps us show that all positive integers have a property given that certain values have the property. More specifically, in a proof by induction, we try to show a statement S holds for a set of integers. There are two main steps to an inductive proof. In the following example, we attempt to show that S holds for all positive integers.

- **Base Case**: First, show that the statement S holds for $n = 1$. In the way S is defined, our base case is $n = 1$. However, the base case doesn't always have to be 1 (it can be $n = 2$ or any other positive integer).

- **Inductive Step**: Next, we assume that the statement S is true for $n = k$, which is our inductive hypothesis. Using this, we try to show that S holds for $n = k+1$. In most induction problems, this is the hardest step.

  Returning to the domino exercise, imagine we have an infinite domino chain. The base case is the first domino being tipped over and the inductive step is the next domino falling over because of the previous domino pushing the next domino. Eventually, all the dominoes will fall down because of the base case and the inductive step.

Note that this method still works if we want to show the validity of S over a set of consecutive positive integers or more generally, even an arithmetic sequence of positive integers.

Now let's try an example.

Example 1. *Prove that the sum of the first n positive integers* $(1+2+\cdots+n)$ *is equal to* $\frac{n(n+1)}{2}$ *for all positive integers* n*.*

Proof. First, we identify the base case, which is $n = 1$. Next, we verify that the statement is true for $n = 1$. It is because we see that $1 = \frac{1\cdot(1+1)}{2}$.

For our second step, we assume that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$, which is our inductive hypothesis, and aim to show that $1 + 2 + \cdots + k + 1 = \frac{(k+1)(k+2)}{2}$. We add $k + 1$ to both sides of the inductive hypothesis.

$$1 + 2 + \cdots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$$

If we put the right hand side over a common denominator, we get $\frac{k(k+1)+2(k+1)}{2}$ which can be factored into $\frac{(k+1)(k+2)}{2}$. Thus $1 + 2 + \cdots + k + 1 = \frac{(k+1)(k+2)}{2}$, and that concludes our proof. □

1.3 Strong Induction

The proof above was an example of **weak induction**. There is also another kind of induction called **strong induction**. [1]

Strong induction is the same as weak induction, except with a twist. These are the two steps of strong induction:

- **Base Case**: This is the same as in weak induction.
- **Inductive Step**: Unlike weak induction, we assume that the statement S is true for all $n = 1, 2, \cdots, k$, which is our inductive hypothesis. Using this, we try to show that S holds for $n = k + 1$. In terms of dominoes, this is like saying that if we know that dominoes $1, 2, \cdots k$ fall, then we know that $k + 1$ must fall.

We will demonstrate how to use strong induction with the following example:

Example 2. *Prove that the nth term of the Fibonacci sequence is*

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Note that we define the Fibonacci sequence as $F_1 = 1, F_2 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$*.*

[1] Note: the names "weak" and "strong" have nothing to do with whether one method is always better. They are both useful for different types of proofs.

Proof. The base case for this example is $n = 3$ (Why? $n = 1$ and $n = 2$ are already defined to be 1). The expression for $n = 3$ is

$$F_3 = \frac{(\frac{1+\sqrt{5}}{2})^3 - (\frac{1-\sqrt{5}}{2})^3}{\sqrt{5}} = 2$$

We'll omit the computation here for the sake of brevity, but you can verify it on your own (hint: use the difference of cubes identity). $F_3 = 1 + 1 = 2$, so the base case works for $n = 3$.

Now for the inductive step. Assume that the expression works for $n = 1, 2, \cdots, k$. In order to prove that the expression holds for $n = k+1$, we must show that it satisfies the definition of the Fibonacci sequence. So we're trying to prove that

$$\frac{(\frac{1+\sqrt{5}}{2})^{k-1} - (\frac{1-\sqrt{5}}{2})^{k-1}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{k} - (\frac{1-\sqrt{5}}{2})^{k}}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}}$$

To do this, we'll take the left hand side and do some algebra:

$$\begin{aligned}
&\frac{(\frac{1+\sqrt{5}}{2})^{k-1} - (\frac{1-\sqrt{5}}{2})^{k-1}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{k} - (\frac{1-\sqrt{5}}{2})^{k}}{\sqrt{5}} \\
=&\frac{(\frac{1+\sqrt{5}}{2})^{k-1} - (\frac{1-\sqrt{5}}{2})^{k-1} + (\frac{1+\sqrt{5}}{2})^{k} - (\frac{1-\sqrt{5}}{2})^{k}}{\sqrt{5}} \\
=&\frac{(\frac{1+\sqrt{5}}{2})^{k-1}(1 + \frac{1+\sqrt{5}}{2}) - (\frac{1-\sqrt{5}}{2})^{k-1}(1 + \frac{1-\sqrt{5}}{2})}{\sqrt{5}}
\end{aligned}$$

Now, $1 + \frac{1+\sqrt{5}}{2}$ happens to equal $(\frac{1+\sqrt{5}}{2})^2$, and $1 + \frac{1-\sqrt{5}}{2} = (1 + \frac{1-\sqrt{5}}{2})^2$ (you can verify yourself), so the expression above becomes

$$\begin{aligned}
&\frac{(\frac{1+\sqrt{5}}{2})^{k-1}(\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^{k-1}(\frac{1-\sqrt{5}}{2})^2}{\sqrt{5}} \\
=&\frac{(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}}{\sqrt{5}}
\end{aligned}$$

So the expression works for $n = k + 1$, and we are done! □

1.4 Problems

Problems are roughly arranged in order of difficulty. Either weak or strong induction should be used in every problem (except for the last problem). Problems marked with a * are challenge problems.

1. Using induction, prove that $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$ holds for all positive integers n.

2. Prove that $1+3+5+\cdots+(2n-1)=n^2$ always holds for all positive integers n.
3. Prove that 2^n+1 is divisible by 3 for all odd positive integers n.
4. Prove $F_1+F_2+...+F_n=F_{n+2}-1$, where F denotes the Fibonacci sequence.
5. Prove $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$
6. Use induction to prove that $1+4+9+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for all integers n, then find the value of $1^2+2^2+3^2+\cdots+100^2$.
7. Prove that $n!>2^n$ for all positive integers $n\geq 4$.
8. Prove that n^2-1 is divisible by 8 for all odd integers n.
9. Show that every integer greater than or equal to 2 can be factored into primes.
10. Prove that for all positive integers n,

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!=(n+1)!-1.$$

11. Show that

$$\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n-1}{n!}=1-\frac{1}{n!}.$$

12. Show that for all positive integers n, n^3+2n is a multiple of 3.
13. Prove that a convex n-gon has $n(n-3)/2$ diagonals.
14. * Show that 2^{k+2} is a divisor of $n^{2^k}-1$ if n is any positive odd integer and k is a positive integer. (Hint: Use the fact that every odd positive integer n can be written as $2m+1$ for some positive integer m).
15. * A chocolate bar consists of unit squares arranged in an $n\times m$ rectangular grid. You may split the bar into individual unit squares, by breaking along the lines. What is the number of breaks required? Prove your answer using induction. *Source: Brilliant*
16. * Prove that every positive integer can be expressed as an alternating-sum of an increasing sequence of powers of 2. For example, $2=2$, $5=1-4+8$, or $8=-8+16$. *Source: Ross Program 2019 Application*
17. * Prove the Binomial Theorem using induction.
18. * We've taken the fact that induction actually works for granted so far. Now try to rigorously prove that proof by induction is valid.

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