Summer Math Circle Handouts

June 9, 2019

1 Solutions to Induction Problems

1. *Proof.* Our base case is $n = 1$. $1 + 2 = 2^{1+1} - 1$, so the base case works.

   For the inductive step, we assume $n = k$ works, so $1+2+2^2+2^3+\cdots+2^k = 2^{k+1}-1$. We add 2^{k+1} to both sides of this equation to get $1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$. The RHS is equivalent to $2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$, so $1 + 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$. We've shown that $n = k + 1$ also works, so we are done. □

2. *Proof.* Let $P(n)$ denote the assertion that $1 + 3 + \cdots + (2n - 1) = n^2$. In this problem, our base case is $n = 1$, which gives $1 = 1^2$, which is true. That is, $P(1)$ is true.

   Now suppose $1 + 3 + \cdots + (2k - 1) = k^2$, that is, $P(k)$ is true. Then, it must also be true that $1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$. Thus, $P(k)$ being true implies $P(k + 1)$ is true and the inductive step is complete, and we are done. □

3. *Proof.* The base is $n = 1$. $2^1 + 1 = 3$, which is divisible by 3, so the base case works.

   Our inductive hypothesis is that 2^k+1 is divisible by 3, where k is an odd number. We are trying to show that the next odd number, $k + 2$, also works. So, we take $2^k + 1$ and multiply it by 4 to get $2^{k+2} + 4$, which is still divisible by 3. If we subtract 3 from this expression to get $2^{k+2}+1$, it is still divisible by 3, so $n = k+2$ works, and we are done. □

4. *Proof.* Clearly $F_1 = 1 = F_3 - 1 = 2 - 1 = 1$, so we have our base case. Now we suppose $F_1 + \cdots + F_n = F_{n+2} - 1$. Adding F_{n+1} to both sides, we see $F_1 + \cdots + F_n + F_{n+1} = F_{n+2} + F_{n+1} - 1 = F_{n+3} - 1$, so the assertion being true for n means it's true for $n + 1$, so the inductive step is complete and we're done. □

5. *Proof.* Our base case is $n = 1$. $\frac{1}{1\cdot 2} = \frac{1}{1+1}$, so the base case works.

   For the inductive step, we assume that $n = k$ works, so $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)} = \frac{k}{k+1}$ works. We want to prove that $n = k$ also works, so we take the equation above and add $\frac{1}{(k+1)(k+2)}$ to both sides:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

If we combine the two fractions on the RHS, we get $\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k+1}{k+2}$. So $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$, and since we've shown that $n=k+1$ works, we are done. □

6. *Proof.* As has been the case, our base case is $n=1$, and our assertion is clearly true, as $1^2=\frac{1\cdot 2\cdot 3}{6}=1$. Now we assume that $1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$. Adding $(k+1)^2$ to both sides, we see that the sum up until $(k+1)^2$ is equal to $\frac{k(k+1)(2k+1)}{6}+(k+1)^2=(k+1)\left(\frac{2k^2+k+6k+6}{6}\right)=(k+1)\cdot\frac{(k+2)(2k+3)}{6}$. □

7. *Proof.* Our base case is $n=4$. $4!=24>2^4=16$, so the base case is true.

   For the inductive hypothesis, assume that $k!>2^k$. It's true that $2\cdot k!>2^k\cdot 2 \implies 2\cdot k!>2^{k+1}$. Since $k+1>2$, we can say that $(k+1)\cdot k!>2^{k+1} \implies (k+1)!>2^{k+1}$, and that completes the proof. □

8. *Proof.* To show a statement is true for odd numbers, note that all odd numbers can be written as $2m+1$. So we will induct on m.

   Our base case is where $m=0$ (or $n=1$). $1^2-1=0$ which is obviously divisible by 8.

   For our inductive step, we assume that $(2m+1)^2-1$ is divisible by 8. We must show that $(2(m+1)+1)^2-1$ is divisible by 8. Expanding and taking the difference between the two quantities, we get that

$$[(2(m+1)+1)^2-1]-[(2m+1)^2-1]=8m+8=8(m+1)$$

   So the difference of the two quantities is a multiple of 8. We know that $(2m+1)^2-1$ is divisible by 8 by the our assumption above. Therefore $(2(m+1)+1)^2-1$ must also be divisible by 8. So we have completed the inductive step, so we are done.

   □

9. *Proof.* The statement $P(n)$ is that an integer n greater than or equal to 2 can be factored into primes.

   Base Case: Prove that the statement holds when $n=2$. We are proving $P(2)$. 2 itself is a prime number, so the prime factorization of 2 is 2. Trivially, the statement $P(2)$ holds.

   Inductive Hypothesis: Assume that for all integers less than or equal to k, the statement holds.

   Inductive Step: Consider the number $k+1$. Case $1:k+1$ is a prime number. When $k+1$ is a prime number, the number is a prime factorization of itself.

Therefore, the statement $P(k+1)$ holds. Case 2 : $k+1$ is not a prime number. We know that $k+1$ is a composite, so $k+1 = p \times q (p, q \in \mathbb{Z}^+)$. Intuitively, we can conclude that p and q are less than or equal to $k+1$. From the induction hypothesis stated above, for all integers less than or equal to k, the statement holds, which means both p and q can be expressed as prime factorizations. In this sense, because $k+1$ is a product of p and q, by multiplying the prime factorizations of p and q, we can get the prime factorization for $k+1$ as well. Therefore, the statement that every integer greater than or equal to 2 can be factored into primes holds for all such integers. □

Credit: Doo San Baik(db478) - Cornell CS 2800 Lectures

10. *Proof.* Our base case is $n = 1$. For that, $1 \cdot 1! = (1+1)! - 1$ so our base case works. For our inductive step, we assume that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1$$

for some positive integer k. Using this inductive assumption, we have

$$\begin{aligned} 1 \cdot 1! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! & \\ &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+2) \cdot (k+1)! - 1 \\ &= (k+2)! - 1. \end{aligned}$$

This completes our inductive setup, so our induction is complete. □

Credit: AoPS Intermediate Algebra

11. *Proof.* Our base case is $n = 1$. $\frac{0}{1!} = 0 = 1 - \frac{1}{1!}$, so the base case works.

Now onto the inductive hypothesis. Assume that

$$\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k-1}{k!} = 1 - \frac{1}{k!}$$

We add $\frac{k}{(k+1)!}$ to each side to get

$$\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k-1}{k!} + \frac{k}{(k+1)!} = 1 - \frac{1}{k!} + \frac{k}{(k+1)!}$$

We can add the two fractions on the RHS to get

$$1 + \frac{-(k+1) + k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

So $n = k+1$ works, and we are done. □

12. *Proof.* Our base case is when $n = 1$. $1^3 + 2(1) = 3$, which is obviously divisible by 3.

For our inductive step, we will assume $k^3 + 2k$ is divisible by 3. We must show that $(ek+1)^3 - 2(k+1)$ is divisible by 3. Expanding and taking the difference of the two quantities, we get

$$\left((k+1)^3 + 2(k+1)\right) - (k^3 + 2k) = 3k + 3 = 3(k+1)$$

Since the difference of the two is divisible by 3 and $k^3 + 2k$ is divisible by 3 (from our inductive hypothesis), $(k+1)^3 - 2(k+1)$ must also be divisible by 3. So we have completed our inductive step, so we are done. □

13. *Proof.* The base case is $n = 3$, a triangle which has 0 diagonals. $3(3-3)/2 = 0$, so the base case works.

    Now, assume that a k-gon has $k(k-3)/2$ diagonals. When we add another vertex to the k-gon, there are $k-2$ vertices that the new vertex can form a diagonal with. In addition, the two vertices next to the new vertex that originally formed a side can also form a diagonal. This gives

    $$k(k-3)/2 + k - 2 + 1 = (k(k-3) + 2k - 2)/2 = (k^2 - k - 2)/2 = (k+1)(k-2)/2$$

    Thus, this expression works for a $k+1$-gon, concluding the proof. □

14. *Proof.* The base case is when $k = 1$. Note that Problem 8 is our base case, so our base case holds. For our inductive hypothesis, we assume that 2^{m+2} is a divisor of $n^{2^m} - 1$ for some odd positive integer n and a positive integer m, and we aim to show that 2^{m+3} is a divisor of $n^{2^{m+1}} - 1$. To do this, we write

    $$\begin{aligned} n^{2^{m+1}} - 1 &= \left(n^{2^m}\right)^2 - 1 \\ &= \left(n^{2^m} - 1\right)\left(n^{2^m} + 1\right). \end{aligned}$$

    Notice that $n^{2^m} - 1$ is divisible by 2^{m+2} by the inductive hypothesis and $n^{2^m} + 1$ is even since n is odd and all powers of odd numbers are odd. Thus, $n^{2^{m+1}} - 1 = \left(n^{2^m} - 1\right)\left(n^{2^m} + 1\right)$ is divisible by $2^{m+2} \cdot 2 = 2^{m+3}$, so we are done. □

15. We will show that the number of breaks needed is $nm - 1$.

    Proof. We start off with our base case.
    Base Case: For a 1×1 square, we are already done, so no steps are needed. $1 \times 1 - 1 = 0$, so the base case is true.

    Inductive Step: Let $P(n, m)$ denote the number of breaks needed to split up an $n \times m$ square. WLOG, we may assume that the first break is along a row, and we get an $n_1 \times m$ and an $n_2 \times m$ bar, where $n_1 + n_2 = n$. By the induction hypothesis, the number of further breaks that we need is $n_1 \times m - 1$ and $n_2 \times m - 1$. Hence, the total number of breaks that we need is

    $$1 + (n_1 \times m - 1) + (n_2 \times m - 1) = (n_1 + n_2) \times m - 1 = n \times m - 1. \square$$

    □

Credit: Brilliant

16. For convenience, denote an alternating-sum of an increasing sequence of powers of 2 that evaluates to a positive integer n as the *2-sum of* n.

Proof. We start off with our base case.
Base case: $k = 1$ can clearly be written as a 2-sum, $1 = 1$.

Inductive step: Assume that $k = 1, 2 \cdots, n$ can be written as a 2-sum. Let 2^m be the least power of two such that $n + 1 \leq 2^m$. If $n + 1 = 2^m$, then that's our 2-sum. Otherwise, we know that $2^m - (n + 1)$ is among the $k = 1, 2 \cdots, n$ that can be written as a 2-sum. Thus, we can flip the signs of the 2-sum of $2^m - (n+1)$ and write $n + 1$ as $n + 1 = -(\text{2-sum of } 2^m m - (n + 1)) + 2^m$.

Therefore, $n + 1$ can be written as a 2-sum, and that concludes the inductive step. □

17. This is the binomial theorem: For any positive integer n,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof. For $n = 1$,

$$(x + y)^1 = x + y = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k$$

Suppose

$$(x + y)^{n-1} = \sum n - 1_{k=0} \binom{n-1}{k} x^{(n-1)-k} y^k$$

Consider $(x+y)^n$.

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\left[\sum_{k=0}^{n-1}\binom{n-1}{k}x^{(n-1)-k}y^k\right] \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{j=0}^{n-1}\binom{n-1}{j}x^{(n-1)-j}y^{j+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{j=0}^{n-1}\binom{n-1}{(j+1)-1}x^{n-(j+1)}y^{j+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k=1}^{n}\binom{n-1}{k-1}x^{n-k}y^k \\
&= \sum_{k=0}^{n}\left[\binom{n-1}{k}x^{n-k}y^k\right] - \binom{n-1}{n}x^0y^n \\
&\quad + \sum_{k=0}^{n}\left[\binom{n-1}{k-1}x^{n-k}y^k\right] - \binom{n-1}{-1}x^ny^0 \\
&= \sum_{k=0}^{n}\left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]x^{n-k}y^k \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k
\end{aligned}$$

and the theorem is proved! □

Credit: Harvey Mudd College Mathematics

18. This is the theorem we are trying to prove:

Theorem. *Let S be a set of positive integers with the following properties:*

(a) The integer 1 belongs to the set.

(b) Whenever the integers $1, 2, 3, \ldots, k$ are in S, the next integer $k+1$ must also be in S.

Then S is the set of all positive integers.

Proof. We will prove this theorem by contradiction.

Let T be the set of all positive integers not in S. By assumption, T is non-empty. Hence it must contain a smallest element, which we will denote by α.

By (a), $0 < \alpha - 1 < \alpha$. Since α is the smallest integer in T, this implies that $1, 2, \ldots, \alpha - 1 \notin T \implies 1, 2, \ldots, \alpha - 1 \in S$.

By (b), S must also contain $(\alpha - 1) + 1 = \alpha$. This contradicts the assumption that $\alpha \subset T$.

Hence set T is empty, and set S contains all positive integers. $\square$ $\square$

Credit: Brilliant

We encourage you to try proving weak induction as well as an exercise.

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